4.5 Exercises

1–6 Evaluate the integral by making the given substitution.

1. $\int \sin \pi x \, dx, \quad u = \pi x$

2. $\int x^3(2 + x^4)^5 \, dx, \quad u = 2 + x^4$

3. $\int x^2\sqrt{x^3 + 1} \, dx, \quad u = x^3 + 1$

4. $\int \frac{dt}{(1 - 6t)^4}, \quad u = 1 - 6t$

5. $\int \cos^3\theta \sin\theta \, d\theta, \quad u = \cos\theta$

6. $\int \frac{\sec^2(1/x)}{x^2} \, dx, \quad u = 1/x$

7–30 Evaluate the indefinite integral.

7. $\int x \sin(x^2) \, dx$

8. $\int x^2 \cos(x^3) \, dx$

9. $\int (1 - 2x)^9 \, dx$

10. $\int (3t + 2)^{2.4} \, dt$

11. $\int (x + 1)\sqrt{2x + x^2} \, dx$

12. $\int \sec^2 2\theta \, d\theta$

13. $\int \sec 3t \tan 3t \, dt$

14. $\int u\sqrt{1 - u^2} \, du$

15. $\int \frac{a + bx^2}{\sqrt{3ax + bx^3}} \, dx$

16. $\int \frac{\sin\sqrt{x}}{\sqrt{x}} \, dx$

17. $\int \sec^2\theta \tan^3\theta \, d\theta$

18. $\int \cos^4\theta \sin\theta \, d\theta$

19. $\int (x^2 + 1)(x^3 + 3x)^4 \, dx$

20. $\int \sqrt{x} \sin(1 + x^{3/2}) \, dx$

21. $\int \frac{\cos x}{\sin^2 x} \, dx$

22. $\int \frac{\cos(\pi/x)}{x^2} \, dx$

23. $\int \frac{z^2}{\sqrt[3]{1 + z^3}} \, dz$

24. $\int \frac{dt}{\cos^2 t\sqrt{1 + \tan t}}$

25. $\int \sqrt{\cot x} \csc^2 x \, dx$

26. $\int \sin t \sec^2(\cos t) \, dt$

27. $\int \sec^3 x \tan x \, dx$

28. $\int x^2\sqrt{2 + x} \, dx$

29. $\int x(2x + 5)^8 \, dx$

30. $\int x^3\sqrt{x^2 + 1} \, dx$

31–34 Evaluate the indefinite integral. Illustrate and check that your answer is reasonable by graphing both the function and its antiderivative (take $C = 0$).

31. $\int x(x^2 - 1)^3 \, dx$

32. $\int \tan^2\theta \sec^2\theta \, d\theta$

33. $\int \sin^3 x \cos x \, dx$

34. $\int \sin x \cos^4 x \, dx$

35–51 Evaluate the definite integral.

35. $\int_0^1 \cos(\pi t/2) \, dt$

36. $\int_0^1 (3t - 1)^{50} \, dt$

37. $\int_0^1 \sqrt[3]{1 + 7x} \, dx$

38. $\int_0^{\sqrt{\pi}} x\cos(x^2) \, dx$

39. $\int_0^{\pi} \sec^2(t/4) \, dt$

40. $\int_{1/6}^{1/2} \csc \pi t \cot \pi t \, dt$

41. $\int_{-\pi/4}^{\pi/4} (x^3 + x^4 \tan x) \, dx$

42. $\int_0^{\pi/2} \cos x \sin(\sin x) \, dx$

43. $\int_0^{13} \frac{dx}{\sqrt[3]{(1 + 2x)^2}}$

44. $\int_0^a x\sqrt{a^2 - x^2} \, dx$

45. $\int_0^a x\sqrt{x^2 + a^2} \, dx \quad (a > 0)$

46. $\int_{-\pi/3}^{\pi/3} x^4 \sin x \, dx$

47. $\int_1^2 x\sqrt{x - 1} \, dx$

48. $\int_0^4 \frac{x}{\sqrt{1 + 2x}} \, dx$

49. $\int_{1/2}^1 \frac{\cos(x^{-2})}{x^3} \, dx$

50. $\int_0^{T/2} \sin(2\pi t/T - \alpha) \, dt$

51. $\int_0^1 \frac{dx}{(1 + \sqrt{x})^4}$

52. Verify that $f(x) = \sin\sqrt[3]{x}$ is an odd function and use that fact to show that

$$0 \leqslant \int_{-2}^{3} \sin\sqrt[3]{x} \, dx \leqslant 1$$

53–54 Use a graph to give a rough estimate of the area of the region that lies under the given curve. Then find the exact area.

53. $y = \sqrt{2x + 1}, \quad 0 \leqslant x \leqslant 1$

54. $y = 2\sin x - \sin 2x, \quad 0 \leqslant x \leqslant \pi$

55. Evaluate $\int_{-2}^{2} (x + 3)\sqrt{4 - x^2} \, dx$ by writing it as a sum of two integrals and interpreting one of those integrals in terms of an area.

56. Evaluate $\int_0^1 x\sqrt{1 - x^4} \, dx$ by making a substitution and interpreting the resulting integral in terms of an area.

57. Breathing is cyclic and a full respiratory cycle from the beginning of inhalation to the end of exhalation takes about 5 s. The maximum rate of air flow into the lungs is about 0.5 L/s. This explains, in part, why the function $f(t) = \frac{1}{2}\sin(2\pi t/5)$ has often been used to model the rate of air flow into the lungs. Use this model to find the volume of inhaled air in the lungs at time t.

58. A model for the basal metabolism rate, in kcal/h, of a young man is $R(t) = 85 - 0.18\cos(\pi t/12)$, where t is the time in hours measured from 5:00 AM. What is the total basal metabolism of this man, $\int_0^{24} R(t) \, dt$, over a 24-hour time period?

59. If f is continuous and $\int_0^4 f(x) \, dx = 10$, find $\int_0^2 f(2x) \, dx$.

60. If f is continuous and $\int_0^9 f(x) \, dx = 4$, find $\int_0^3 xf(x^2) \, dx$.

61. If f is continuous on $\mathbb{R}$, prove that

$$\int_a^b f(-x)\,dx = \int_{-b}^{-a} f(x)\,dx$$

For the case where $f(x) \geqslant 0$ and $0 < a < b$, draw a diagram to interpret this equation geometrically as an equality of areas.

62. If f is continuous on $\mathbb{R}$, prove that

$$\int_a^b f(x+c)\,dx = \int_{a+c}^{b+c} f(x)\,dx$$

For the case where $f(x) \geqslant 0$, draw a diagram to interpret this equation geometrically as an equality of areas.

63. If a and b are positive numbers, show that

$$\int_0^1 x^a(1-x)^b\,dx = \int_0^1 x^b(1-x)^a\,dx$$

64. If f is continuous on $[0, \pi]$, use the substitution $u = \pi - x$ to show that

$$\int_0^\pi x f(\sin x)\,dx = \frac{\pi}{2}\int_0^\pi f(\sin x)\,dx$$

65. If f is continuous, prove that

$$\int_0^{\pi/2} f(\cos x)\,dx = \int_0^{\pi/2} f(\sin x)\,dx$$

66. Use Exercise 65 to evaluate $\int_0^{\pi/2} \cos^2 x\,dx$ and $\int_0^{\pi/2} \sin^2 x\,dx$.

The following exercises are intended only for those who have already covered Chapter 6.

67–84 Evaluate the integral.

67. $\displaystyle\int \frac{dx}{5-3x}$

68. $\displaystyle\int e^x \sin(e^x)\,dx$

69. $\displaystyle\int \frac{(\ln x)^2}{x}\,dx$

70. $\displaystyle\int \frac{dx}{ax+b} \quad (a \neq 0)$

71. $\displaystyle\int e^x\sqrt{1+e^x}\,dx$

72. $\displaystyle\int e^{\cos t}\sin t\,dt$

73. $\displaystyle\int e^{\tan x}\sec^2 x\,dx$

74. $\displaystyle\int \frac{\tan^{-1}x}{1+x^2}\,dx$

75. $\displaystyle\int \frac{1+x}{1+x^2}\,dx$

76. $\displaystyle\int \frac{\sin(\ln x)}{x}\,dx$

77. $\displaystyle\int \frac{\sin 2x}{1+\cos^2 x}\,dx$

78. $\displaystyle\int \frac{\sin x}{1+\cos^2 x}\,dx$

79. $\displaystyle\int \cot x\,dx$

80. $\displaystyle\int \frac{x}{1+x^4}\,dx$

81. $\displaystyle\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}}$

82. $\displaystyle\int_0^1 xe^{-x^2}\,dx$

83. $\displaystyle\int_0^1 \frac{e^z+1}{e^z+z}\,dz$

84. $\displaystyle\int_0^{1/2} \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx$

85. Use Exercise 64 to evaluate the integral

$$\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx$$

4 Review

Concept Check

1. (a) Write an expression for a Riemann sum of a function f. Explain the meaning of the notation that you use.
(b) If $f(x) \geqslant 0$, what is the geometric interpretation of a Riemann sum? Illustrate with a diagram.
(c) If $f(x)$ takes on both positive and negative values, what is the geometric interpretation of a Riemann sum? Illustrate with a diagram.

2. (a) Write the definition of the definite integral of a continuous function from a to b.
(b) What is the geometric interpretation of $\int_a^b f(x)\,dx$ if $f(x) \geqslant 0$?
(c) What is the geometric interpretation of $\int_a^b f(x)\,dx$ if $f(x)$ takes on both positive and negative values? Illustrate with a diagram.

3. State both parts of the Fundamental Theorem of Calculus.

4. (a) State the Net Change Theorem.
(b) If $r(t)$ is the rate at which water flows into a reservoir, what does $\int_{t_1}^{t_2} r(t)\,dt$ represent?

5. Suppose a particle moves back and forth along a straight line with velocity $v(t)$, measured in feet per second, and acceleration $a(t)$.
(a) What is the meaning of $\int_{60}^{120} v(t)\,dt$?
(b) What is the meaning of $\int_{60}^{120} |v(t)|\,dt$?
(c) What is the meaning of $\int_{60}^{120} a(t)\,dt$?

6. (a) Explain the meaning of the indefinite integral $\int f(x)\,dx$.
(b) What is the connection between the definite integral $\int_a^b f(x)\,dx$ and the indefinite integral $\int f(x)\,dx$?

7. Explain exactly what is meant by the statement that "differentiation and integration are inverse processes."

8. State the Substitution Rule. In practice, how do you use it?

True-False Quiz

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. If f and g are continuous on $[a, b]$, then
$$\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

2. If f and g are continuous on $[a, b]$, then
$$\int_a^b [f(x)g(x)]\,dx = \left(\int_a^b f(x)\,dx\right)\left(\int_a^b g(x)\,dx\right)$$

3. If f is continuous on $[a, b]$, then
$$\int_a^b 5f(x)\,dx = 5\int_a^b f(x)\,dx$$

4. If f is continuous on $[a, b]$, then
$$\int_a^b xf(x)\,dx = x\int_a^b f(x)\,dx$$

5. If f is continuous on $[a, b]$ and $f(x) \geqslant 0$, then
$$\int_a^b \sqrt{f(x)}\,dx = \sqrt{\int_a^b f(x)\,dx}$$

6. If f' is continuous on $[1, 3]$, then $\int_1^3 f'(v)\,dv = f(3) - f(1)$.

7. If f and g are continuous and $f(x) \geqslant g(x)$ for $a \leqslant x \leqslant b$, then
$$\int_a^b f(x)\,dx \geqslant \int_a^b g(x)\,dx$$

8. If f and g are differentiable and $f(x) \geqslant g(x)$ for $a < x < b$, then $f'(x) \geqslant g'(x)$ for $a < x < b$.

9. $\displaystyle\int_{-1}^1 \left(x^5 - 6x^9 + \frac{\sin x}{(1 + x^4)^2}\right) dx = 0$

10. $\displaystyle\int_{-5}^5 (ax^2 + bx + c)\,dx = 2\int_0^5 (ax^2 + c)\,dx$

11. All continuous functions have derivatives.

12. All continuous functions have antiderivatives.

13. $\displaystyle\int_\pi^{2\pi} \frac{\sin x}{x}\,dx = \int_\pi^{3\pi} \frac{\sin x}{x}\,dx + \int_{3\pi}^{2\pi} \frac{\sin x}{x}\,dx$

14. If $\int_0^1 f(x)\,dx = 0$, then $f(x) = 0$ for $0 \leqslant x \leqslant 1$.

15. If f is continuous on $[a, b]$, then
$$\frac{d}{dx}\left(\int_a^b f(x)\,dx\right) = f(x)$$

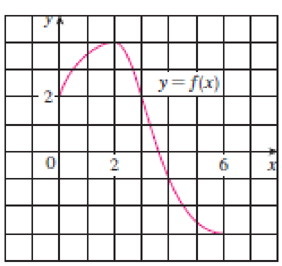
16. $\int_0^2 (x - x^3)\,dx$ represents the area under the curve $y = x - x^3$ from 0 to 2.

17. $\displaystyle\int_{-2}^1 \frac{1}{x^4}\,dx = -\frac{3}{8}$

18. If f has a discontinuity at 0, then $\int_{-1}^1 f(x)\,dx$ does not exist.

Exercises

1. Use the given graph of f to find the Riemann sum with six subintervals. Take the sample points to be (a) left endpoints and (b) midpoints. In each case draw a diagram and explain what the Riemann sum represents.



2. (a) Evaluate the Riemann sum for
$$f(x) = x^2 - x \qquad 0 \leqslant x \leqslant 2$$
with four subintervals, taking the sample points to be right endpoints. Explain, with the aid of a diagram, what the Riemann sum represents.
(b) Use the definition of a definite integral (with right endpoints) to calculate the value of the integral
$$\int_0^2 (x^2 - x)\,dx$$
(c) Use the Fundamental Theorem to check your answer to part (b).
(d) Draw a diagram to explain the geometric meaning of the integral in part (b).

3. Evaluate

$$\int_0^1 \left(x + \sqrt{1 - x^2}\right) dx$$

by interpreting it in terms of areas.

4. Express

$$\lim_{n \to \infty} \sum_{i=1}^{n} \sin x_i \, \Delta x$$

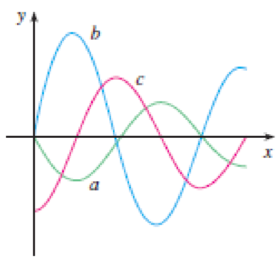
as a definite integral on the interval $[0, \pi]$ and then evaluate the integral.

5. If $\int_0^6 f(x)\,dx = 10$ and $\int_0^4 f(x)\,dx = 7$, find $\int_4^6 f(x)\,dx$.

CAS **6.** (a) Write $\int_1^5 (x + 2x^5)\,dx$ as a limit of Riemann sums, taking the sample points to be right endpoints. Use a computer algebra system to evaluate the sum and to compute the limit.

(b) Use the Fundamental Theorem to check your answer to part (a).

7. The following figure shows the graphs of f, f', and $\int_0^x f(t)\,dt$. Identify each graph, and explain your choices.



8. Evaluate:

(a) $\displaystyle\int_0^{\pi/2} \frac{d}{dx}\left(\sin\frac{x}{2}\cos\frac{x}{3}\right) dx$

(b) $\displaystyle\frac{d}{dx}\int_0^{\pi/2} \sin\frac{x}{2}\cos\frac{x}{3}\,dx$

(c) $\displaystyle\frac{d}{dx}\int_x^{\pi/2} \sin\frac{t}{2}\cos\frac{t}{3}\,dt$

9–28 Evaluate the integral.

9. $\int_1^2 (8x^3 + 3x^2)\,dx$

10. $\int_0^T (x^4 - 8x + 7)\,dx$

11. $\int_0^1 (1 - x^9)\,dx$

12. $\int_0^1 (1 - x)^9\,dx$

13. $\displaystyle\int_1^9 \frac{\sqrt{u} - 2u^2}{u}\,du$

14. $\int_0^1 \left(\sqrt[4]{u} + 1\right)^2 du$

15. $\int_0^1 y(y^2 + 1)^5\,dy$

16. $\int_0^2 y^2\sqrt{1 + y^3}\,dy$

17. $\displaystyle\int_1^5 \frac{dt}{(t - 4)^2}$

18. $\int_0^1 \sin(3\pi t)\,dt$

19. $\int_0^1 v^2\cos(v^3)\,dv$

20. $\displaystyle\int_{-1}^1 \frac{\sin x}{1 + x^2}\,dx$

21. $\displaystyle\int_{-\pi/4}^{\pi/4} \frac{t^4 \tan t}{2 + \cos t}\,dt$

22. $\displaystyle\int \frac{x + 2}{\sqrt{x^2 + 4x}}\,dx$

23. $\int \sin \pi t \cos \pi t\,dt$

24. $\int \sin x \cos(\cos x)\,dx$

25. $\int_0^{\pi/8} \sec 2\theta \tan 2\theta\,d\theta$

26. $\int_0^{\pi/4} (1 + \tan t)^3 \sec^2 t\,dt$

27. $\int_0^3 |x^2 - 4|\,dx$

28. $\int_0^4 |\sqrt{x} - 1|\,dx$

29–30 Evaluate the indefinite integral. Illustrate and check that your answer is reasonable by graphing both the function and its antiderivative (take $C = 0$).

29. $\displaystyle\int \frac{\cos x}{\sqrt{1 + \sin x}}\,dx$

30. $\displaystyle\int \frac{x^3}{\sqrt{x^2 + 1}}\,dx$

31. Use a graph to give a rough estimate of the area of the region that lies under the curve $y = x\sqrt{x}$, $0 \leqslant x \leqslant 4$. Then find the exact area.

32. Graph the function $f(x) = \cos^2 x \sin x$ and use the graph to guess the value of the integral $\int_0^{2\pi} f(x)\,dx$. Then evaluate the integral to confirm your guess.

33–38 Find the derivative of the function.

33. $\displaystyle F(x) = \int_0^x \frac{t^2}{1 + t^3}\,dt$

34. $\displaystyle F(x) = \int_x^1 \sqrt{t + \sin t}\,dt$

35. $\displaystyle g(x) = \int_0^{x^4} \cos(t^2)\,dt$

36. $\displaystyle g(x) = \int_1^{\sin x} \frac{1 - t^2}{1 + t^4}\,dt$

37. $\displaystyle y = \int_{\sqrt{x}}^x \frac{\cos\theta}{\theta}\,d\theta$

38. $\displaystyle y = \int_{2x}^{3x+1} \sin(t^4)\,dt$

39–40 Use Property 8 of integrals to estimate the value of the integral.

39. $\int_1^3 \sqrt{x^2 + 3}\,dx$

40. $\displaystyle\int_3^5 \frac{1}{x + 1}\,dx$

41–42 Use the properties of integrals to verify the inequality.

41. $\displaystyle\int_0^1 x^2 \cos x\,dx \leqslant \frac{1}{3}$

42. $\displaystyle\int_{\pi/4}^{\pi/2} \frac{\sin x}{x}\,dx \leqslant \frac{\sqrt{2}}{2}$

43. Use the Midpoint Rule with $n = 6$ to approximate $\int_0^3 \sin(x^3)\,dx$.

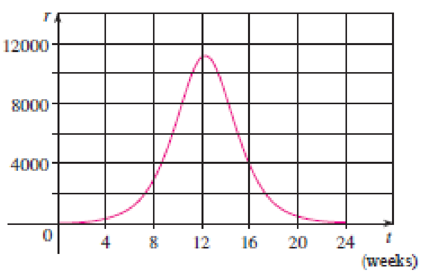
44. A particle moves along a line with velocity function $v(t) = t^2 - t$, where v is measured in meters per second. Find (a) the displacement and (b) the distance traveled by the particle during the time interval $[0, 5]$.

45. Let $r(t)$ be the rate at which the world's oil is consumed, where t is measured in years starting at $t = 0$ on January 1, 2000, and $r(t)$ is measured in barrels per year. What does $\int_0^8 r(t)\,dt$ represent?

46. A radar gun was used to record the speed of a runner at the times given in the table. Use the Midpoint Rule to estimate the distance the runner covered during those 5 seconds.

t (s)	v (m/s)	t (s)	v (m/s)
0	0	3.0	10.51
0.5	4.67	3.5	10.67
1.0	7.34	4.0	10.76
1.5	8.86	4.5	10.81
2.0	9.73	5.0	10.81
2.5	10.22		

47. A population of honeybees increased at a rate of $r(t)$ bees per week, where the graph of r is as shown. Use the Midpoint Rule with six subintervals to estimate the increase in the bee population during the first 24 weeks.



48. Let

$$f(x) = \begin{cases} -x - 1 & \text{if } -3 \leq x \leq 0 \\ -\sqrt{1 - x^2} & \text{if } 0 \leq x \leq 1 \end{cases}$$

Evaluate $\int_{-3}^{1} f(x)\,dx$ by interpreting the integral as a difference of areas.

49. If f is continuous and $\int_0^2 f(x)\,dx = 6$, evaluate $\int_0^{\pi/2} f(2\sin\theta)\cos\theta\,d\theta$.

50. The Fresnel function $S(x) = \int_0^x \sin\left(\frac{1}{2}\pi t^2\right) dt$ was introduced in Section 4.3. Fresnel also used the function

$$C(x) = \int_0^x \cos\left(\tfrac{1}{2}\pi t^2\right) dt$$

in his theory of the diffraction of light waves.

(a) On what intervals is C increasing?

(b) On what intervals is C concave upward?

CAS (c) Use a graph to solve the following equation correct to two decimal places:

$$\int_0^x \cos\left(\tfrac{1}{2}\pi t^2\right) dt = 0.7$$

CAS (d) Plot the graphs of C and S on the same screen. How are these graphs related?

51. If f is a continuous function such that

$$\int_0^x f(t)\,dt = x\sin x + \int_0^x \frac{f(t)}{1 + t^2}\,dt$$

for all x, find an explicit formula for $f(x)$.

52. Find a function f and a value of the constant a such that

$$2\int_a^x f(t)\,dt = 2\sin x - 1$$

53. If f' is continuous on $[a, b]$, show that

$$2\int_a^b f(x)\,f'(x)\,dx = [f(b)]^2 - [f(a)]^2$$

54. Find $\displaystyle\lim_{h \to 0} \frac{1}{h} \int_2^{2+h} \sqrt{1 + t^3}\,dt$.

55. If f is continuous on $[0, 1]$, prove that

$$\int_0^1 f(x)\,dx = \int_0^1 f(1 - x)\,dx$$

56. Evaluate

$$\lim_{n \to \infty} \frac{1}{n}\left[\left(\frac{1}{n}\right)^9 + \left(\frac{2}{n}\right)^9 + \left(\frac{3}{n}\right)^9 + \cdots + \left(\frac{n}{n}\right)^9\right]$$