4.2 Exercises

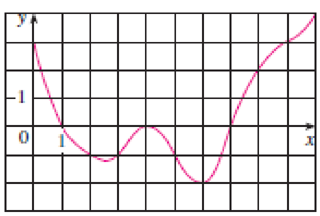
1. Evaluate the Riemann sum for $f(x) = 3 - \frac{1}{2}x$, $2 \leqslant x \leqslant 14$, with six subintervals, taking the sample points to be left endpoints. Explain, with the aid of a diagram, what the Riemann sum represents.

2. If $f(x) = x^2 - 2x$, $0 \leqslant x \leqslant 3$, evaluate the Riemann sum with $n = 6$, taking the sample points to be right endpoints. What does the Riemann sum represent? Illustrate with a diagram.

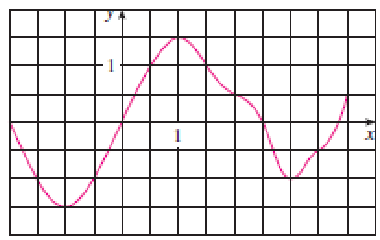
3. If $f(x) = \sqrt{x} - 2$, $1 \leqslant x \leqslant 6$, find the Riemann sum with $n = 5$ correct to six decimal places, taking the sample points to be midpoints. What does the Riemann sum represent? Illustrate with a diagram.

4. (a) Find the Riemann sum for $f(x) = \sin x$, $0 \leqslant x \leqslant 3\pi/2$, with six terms, taking the sample points to be right endpoints. (Give your answer correct to six decimal places.) Explain what the Riemann sum represents with the aid of a sketch.
(b) Repeat part (a) with midpoints as the sample points.

5. The graph of a function f is given. Estimate $\int_0^{10} f(x)\,dx$ using five subintervals with (a) right endpoints, (b) left endpoints, and (c) midpoints.



6. The graph of g is shown. Estimate $\int_{-2}^{4} g(x)\,dx$ with six subintervals using (a) right endpoints, (b) left endpoints, and (c) midpoints.



7. A table of values of an increasing function f is shown. Use the table to find lower and upper estimates for $\int_{10}^{30} f(x)\,dx$.

x	10	14	18	22	26	30
$f(x)$	-12	-6	-2	1	3	8

8. The table gives the values of a function obtained from an experiment. Use them to estimate $\int_3^9 f(x)\,dx$ using three equal subintervals with (a) right endpoints, (b) left endpoints, and (c) midpoints. If the function is known to be an increasing function, can you say whether your estimates are less than or greater than the exact value of the integral?

x	3	4	5	6	7	8	9
$f(x)$	-3.4	-2.1	-0.6	0.3	0.9	1.4	1.8

9–12 Use the Midpoint Rule with the given value of n to approximate the integral. Round the answer to four decimal places.

9. $\int_0^8 \sin\sqrt{x}\,dx, \quad n = 4$

10. $\int_0^{\pi/2} \cos^4 x\,dx, \quad n = 4$

11. $\int_0^2 \frac{x}{x+1}\,dx, \quad n = 5$

12. $\int_1^4 \sqrt{x^3+1}\,dx, \quad n = 6$

CAS **13.** If you have a CAS that evaluates midpoint approximations and graphs the corresponding rectangles (use `RiemannSum` or `middlesum` and `middlebox` commands in Maple), check the answer to Exercise 11 and illustrate with a graph. Then repeat with $n = 10$ and $n = 20$.

14. With a programmable calculator or computer (see the instructions for Exercise 9 in Section 4.1), compute the left and right Riemann sums for the function $f(x) = x/(x+1)$ on the interval $[0, 2]$ with $n = 100$. Explain why these estimates show that

$$0.8946 < \int_0^2 \frac{x}{x+1}\,dx < 0.9081$$

15. Use a calculator or computer to make a table of values of right Riemann sums R_n for the integral $\int_0^\pi \sin x\,dx$ with $n = 5, 10, 50,$ and 100. What value do these numbers appear to be approaching?

16. Use a calculator or computer to make a table of values of left and right Riemann sums L_n and R_n for the integral $\int_0^2 \sqrt{1+x^4}\,dx$ with $n = 5, 10, 50,$ and 100. Between what two numbers must the value of the integral lie? Can you make a similar statement for the integral $\int_{-1}^2 \sqrt{1+x^4}\,dx$? Explain.

17–20 Express the limit as a definite integral on the given interval.

17. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1-x_i^2}{4+x_i^2}\,\Delta x, \quad [2, 6]$

18. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\cos x_i}{x_i}\,\Delta x, \quad [\pi, 2\pi]$

19. $\lim_{n\to\infty} \sum_{i=1}^{n} [5(x_i^*)^3 - 4x_i^*]\,\Delta x, \quad [2, 7]$

20. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{x_i^*}{(x_i^*)^2+4}\,\Delta x, \quad [1, 3]$

21–25 Use the form of the definition of the integral given in Theorem 4 to evaluate the integral.

21. $\int_2^5 (4 - 2x)\,dx$

22. $\int_1^4 (x^2 - 4x + 2)\,dx$

23. $\int_{-2}^0 (x^2 + x)\,dx$

24. $\int_0^2 (2x - x^3)\,dx$

25. $\int_0^1 (x^3 - 3x^2)\,dx$

26. (a) Find an approximation to the integral $\int_0^4 (x^2 - 3x)\,dx$ using a Riemann sum with right endpoints and $n = 8$.
(b) Draw a diagram like Figure 3 to illustrate the approximation in part (a).
(c) Use Theorem 4 to evaluate $\int_0^4 (x^2 - 3x)\,dx$.
(d) Interpret the integral in part (c) as a difference of areas and illustrate with a diagram like Figure 4.

27. Prove that $\int_a^b x\,dx = \frac{b^2 - a^2}{2}$.

28. Prove that $\int_a^b x^2\,dx = \frac{b^3 - a^3}{3}$.

29–30 Express the integral as a limit of Riemann sums. Do not evaluate the limit.

29. $\int_2^6 \frac{x}{1+x^5}\,dx$

30. $\int_0^{2\pi} x^2 \sin x\,dx$

CAS **31–32** Express the integral as a limit of sums. Then evaluate, using a computer algebra system to find both the sum and the limit.

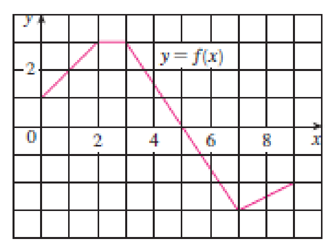
31. $\int_0^\pi \sin 5x\,dx$

32. $\int_2^{10} x^6\,dx$

33. The graph of f is shown. Evaluate each integral by interpreting it in terms of areas.

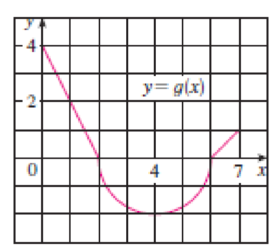
(a) $\int_0^2 f(x)\,dx$ (b) $\int_0^5 f(x)\,dx$

(c) $\int_5^7 f(x)\,dx$ (d) $\int_0^9 f(x)\,dx$



34. The graph of g consists of two straight lines and a semicircle. Use it to evaluate each integral.

(a) $\int_0^2 g(x)\,dx$ (b) $\int_2^6 g(x)\,dx$ (c) $\int_0^7 g(x)\,dx$



35–40 Evaluate the integral by interpreting it in terms of areas.

35. $\int_{-1}^{2} (1 - x)\,dx$

36. $\int_{0}^{9} \left(\frac{1}{3}x - 2\right)\,dx$

37. $\int_{-3}^{0} \left(1 + \sqrt{9 - x^2}\right)\,dx$

38. $\int_{-5}^{5} \left(x - \sqrt{25 - x^2}\right)\,dx$

39. $\int_{-1}^{2} |\,x\,|\,dx$

40. $\int_{0}^{10} |\,x - 5\,|\,dx$

41. Evaluate $\int_{\pi}^{\pi} \sin^2 x \cos^4 x\,dx$.

42. Given that $\int_0^1 3x\sqrt{x^2 + 4}\,dx = 5\sqrt{5} - 8$, what is $\int_1^0 3u\sqrt{u^2 + 4}\,du$?

43. In Example 2 in Section 4.1 we showed that $\int_0^1 x^2\,dx = \frac{1}{3}$. Use this fact and the properties of integrals to evaluate $\int_0^1 (5 - 6x^2)\,dx$.

44. Use the properties of integrals and the result of Example 3 to evaluate $\int_2^5 (1 + 3x^4)\,dx$.

45. Use the results of Exercises 27 and 28 and the properties of integrals to evaluate $\int_1^4 (2x^2 - 3x + 1)\,dx$.

46. Use the result of Exercise 27 and the fact that $\int_0^{\pi/2} \cos x\,dx = 1$ (from Exercise 29 in Section 4.1), together with the properties of integrals, to evaluate $\int_0^{\pi/2} (2\cos x - 5x)\,dx$.

47. Write as a single integral in the form $\int_a^b f(x)\,dx$:

$$\int_{-2}^{2} f(x)\,dx + \int_{2}^{5} f(x)\,dx - \int_{-2}^{-1} f(x)\,dx$$

48. If $\int_1^5 f(x)\,dx = 12$ and $\int_4^5 f(x)\,dx = 3.6$, find $\int_1^4 f(x)\,dx$.

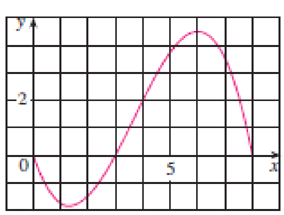
49. If $\int_0^9 f(x)\,dx = 37$ and $\int_0^9 g(x)\,dx = 16$, find $\int_0^9 [2f(x) + 3g(x)]\,dx$.

50. Find $\int_0^5 f(x)\,dx$ if

$$f(x) = \begin{cases} 3 & \text{for } x < 3 \\ x & \text{for } x \geq 3 \end{cases}$$

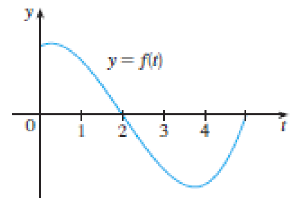
51. For the function f whose graph is shown, list the following quantities in increasing order, from smallest to largest, and explain your reasoning.

(A) $\int_0^8 f(x)\,dx$
(B) $\int_0^3 f(x)\,dx$
(C) $\int_3^8 f(x)\,dx$
(D) $\int_4^8 f(x)\,dx$
(E) $f'(1)$



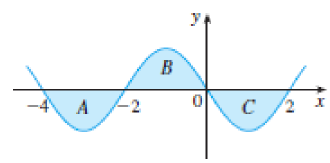
52. If $F(x) = \int_2^x f(t)\,dt$, where f is the function whose graph is given, which of the following values is largest?

(A) $F(0)$
(B) $F(1)$
(C) $F(2)$
(D) $F(3)$
(E) $F(4)$



53. Each of the regions A, B, and C bounded by the graph of f and the x-axis has area 3. Find the value of

$$\int_{-4}^{2} [\,f(x) + 2x + 5\,]\,dx$$



54. Suppose f has absolute minimum value m and absolute maximum value M. Between what two values must $\int_0^2 f(x)\,dx$ lie? Which property of integrals allows you to make your conclusion?

55–58 Use the properties of integrals to verify the inequality without evaluating the integrals.

55. $\int_0^4 (x^2 - 4x + 4)\,dx \geq 0$

56. $\int_0^1 \sqrt{1 + x^2}\,dx \leq \int_0^1 \sqrt{1 + x}\,dx$

57. $2 \leq \int_{-1}^{1} \sqrt{1 + x^2}\,dx \leq 2\sqrt{2}$

58. $\dfrac{\sqrt{2}\,\pi}{24} \leq \int_{\pi/6}^{\pi/4} \cos x\,dx \leq \dfrac{\sqrt{3}\,\pi}{24}$

59–64 Use Property 8 to estimate the value of the integral.

59. $\int_1^4 \sqrt{x}\,dx$

60. $\int_0^2 \dfrac{1}{1 + x^2}\,dx$

61. $\int_{\pi/4}^{\pi/3} \tan x\,dx$

62. $\int_0^2 (x^3 - 3x + 3)\,dx$

63. $\int_{-1}^{1} \sqrt{1 + x^4}\,dx$

64. $\int_{\pi}^{2\pi} (x - 2\sin x)\,dx$

65–66 Use properties of integrals, together with Exercises 27 and 28, to prove the inequality.

65. $\int_1^3 \sqrt{x^4 + 1}\, dx \geq \frac{26}{3}$

66. $\int_0^{\pi/2} x \sin x\, dx \leq \frac{\pi^2}{8}$

67. Prove Property 3 of integrals.

68. (a) If f is continuous on $[a, b]$, show that

$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx$$

[*Hint:* $-|f(x)| \leq f(x) \leq |f(x)|$.]

(b) Use the result of part (a) to show that

$$\left| \int_0^{2\pi} f(x) \sin 2x\, dx \right| \leq \int_0^{2\pi} |f(x)|\, dx$$

69. Let $f(x) = 0$ if x is any rational number and $f(x) = 1$ if x is any irrational number. Show that f is not integrable on $[0, 1]$.

70. Let $f(0) = 0$ and $f(x) = 1/x$ if $0 < x \leq 1$. Show that f is not integrable on $[0, 1]$. [*Hint:* Show that the first term in the Riemann sum, $f(x_1^*)\, \Delta x$, can be made arbitrarily large.]

71–72 Express the limit as a definite integral.

71. $\lim_{n \to \infty} \sum_{i=1}^{n} \frac{i^4}{n^5}$ [*Hint:* Consider $f(x) = x^4$.]

72. $\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{1 + (i/n)^2}$

73. Find $\int_1^2 x^{-2}\, dx$. *Hint:* Choose x_i^* to be the geometric mean of x_{i-1} and x_i (that is, $x_i^* = \sqrt{x_{i-1} x_i}$) and use the identity

$$\frac{1}{m(m+1)} = \frac{1}{m} - \frac{1}{m+1}$$

DISCOVERY PROJECT AREA FUNCTIONS

1. (a) Draw the line $y = 2t + 1$ and use geometry to find the area under this line, above the t-axis, and between the vertical lines $t = 1$ and $t = 3$.

(b) If $x > 1$, let $A(x)$ be the area of the region that lies under the line $y = 2t + 1$ between $t = 1$ and $t = x$. Sketch this region and use geometry to find an expression for $A(x)$.

(c) Differentiate the area function $A(x)$. What do you notice?

2. (a) If $x \geq -1$, let

$$A(x) = \int_{-1}^{x} (1 + t^2)\, dt$$

$A(x)$ represents the area of a region. Sketch that region.

(b) Use the result of Exercise 28 in Section 4.2 to find an expression for $A(x)$.

(c) Find $A'(x)$. What do you notice?

(d) If $x \geq -1$ and h is a small positive number, then $A(x + h) - A(x)$ represents the area of a region. Describe and sketch the region.

(e) Draw a rectangle that approximates the region in part (d). By comparing the areas of these two regions, show that

$$\frac{A(x + h) - A(x)}{h} \approx 1 + x^2$$

(f) Use part (e) to give an intuitive explanation for the result of part (c).

3. (a) Draw the graph of the function $f(x) = \cos(x^2)$ in the viewing rectangle $[0, 2]$ by $[-1.25, 1.25]$.

(b) If we define a new function g by

$$g(x) = \int_0^x \cos(t^2)\, dt$$

then $g(x)$ is the area under the graph of f from 0 to x [until $f(x)$ becomes negative, at which point $g(x)$ becomes a difference of areas]. Use part (a) to determine the value of

Graphing calculator or computer required

x at which $g(x)$ starts to decrease. [Unlike the integral in Problem 2, it is impossible to evaluate the integral defining g to obtain an explicit expression for $g(x)$.]

(c) Use the integration command on your calculator or computer to estimate $g(0.2)$, $g(0.4)$, $g(0.6)$, . . . , $g(1.8)$, $g(2)$. Then use these values to sketch a graph of g.

(d) Use your graph of g from part (c) to sketch the graph of g' using the interpretation of $g'(x)$ as the slope of a tangent line. How does the graph of g' compare with the graph of f?

4. Suppose f is a continuous function on the interval $[a, b]$ and we define a new function g by the equation

$$g(x) = \int_a^x f(t)\, dt$$

Based on your results in Problems 1–3, conjecture an expression for $g'(x)$.