3.9 Exercises

1–18 Find the most general antiderivative of the function. (Check your answer by differentiation.)

1. $f(x) = x - 3$

2. $f(x) = \frac{1}{2}x^2 - 2x + 6$

3. $f(x) = \frac{1}{2} + \frac{3}{4}x^2 - \frac{4}{5}x^3$

4. $f(x) = 8x^9 - 3x^6 + 12x^3$

5. $f(x) = (x + 1)(2x - 1)$

6. $f(x) = x(2 - x)^2$

7. $f(x) = 7x^{2/5} + 8x^{-4/5}$

8. $f(x) = x^{3.4} - 2x^{\sqrt{2}-1}$

9. $f(x) = \sqrt{2}$

10. $f(x) = \pi^2$

11. $f(x) = \dfrac{10}{x^9}$

12. $g(x) = \dfrac{5 - 4x^3 + 2x^6}{x^6}$

13. $g(t) = \dfrac{1 + t + t^2}{\sqrt{t}}$

14. $f(t) = 3\cos t - 4\sin t$

15. $h(\theta) = 2\sin\theta - \sec^2\theta$

16. $f(\theta) = 6\theta^2 - 7\sec^2\theta$

17. $f(t) = 2\sec t\tan t + \frac{1}{2}t^{-1/2}$

18. $f(x) = 2\sqrt{x} + 6\cos x$

19–20 Find the antiderivative F of f that satisfies the given condition. Check your answer by comparing the graphs of f and F.

19. $f(x) = 5x^4 - 2x^5$, $F(0) = 4$

20. $f(x) = x + 2\sin x$, $F(0) = -6$

21–40 Find f.

21. $f''(x) = 20x^3 - 12x^2 + 6x$

22. $f''(x) = x^6 - 4x^4 + x + 1$

23. $f''(x) = \frac{2}{3}x^{2/3}$

24. $f''(x) = 6x + \sin x$

25. $f'''(t) = \cos t$

26. $f'''(t) = t - \sqrt{t}$

27. $f'(x) = 1 + 3\sqrt{x}$, $f(4) = 25$

28. $f'(x) = 5x^4 - 3x^2 + 4$, $f(-1) = 2$

29. $f'(x) = \sqrt{x}\,(6 + 5x)$, $f(1) = 10$

30. $f'(t) = t + 1/t^3$, $t > 0$, $f(1) = 6$

31. $f'(t) = 2\cos t + \sec^2 t$, $-\pi/2 < t < \pi/2$, $f(\pi/3) = 4$

32. $f'(x) = x^{-1/3}$, $f(1) = 1$, $f(-1) = -1$

33. $f''(x) = -2 + 12x - 12x^2$, $f(0) = 4$, $f'(0) = 12$

34. $f''(x) = 8x^3 + 5$, $f(1) = 0$, $f'(1) = 8$

35. $f''(\theta) = \sin\theta + \cos\theta$, $f(0) = 3$, $f'(0) = 4$

36. $f''(t) = 3/\sqrt{t}$, $f(4) = 20$, $f'(4) = 7$

37. $f''(x) = 4 + 6x + 24x^2$, $f(0) = 3$, $f(1) = 10$

38. $f''(x) = 20x^3 + 12x^2 + 4$, $f(0) = 8$, $f(1) = 5$

39. $f''(x) = 2 + \cos x$, $f(0) = -1$, $f(\pi/2) = 0$

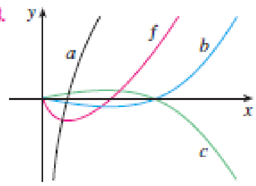
40. $f'''(x) = \cos x$, $f(0) = 1$, $f'(0) = 2$, $f''(0) = 3$

41. Given that the graph of f passes through the point $(1, 6)$ and that the slope of its tangent line at $(x, f(x))$ is $2x + 1$, find $f(2)$.

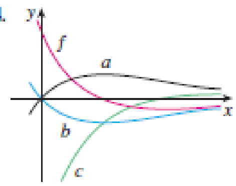
42. Find a function f such that $f'(x) = x^3$ and the line $x + y = 0$ is tangent to the graph of f.

43–44 The graph of a function f is shown. Which graph is an antiderivative of f and why?

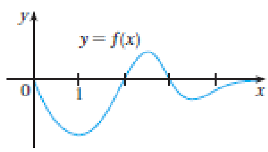
43.



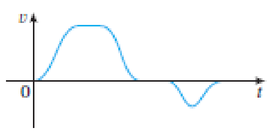
44.



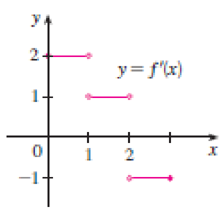
45. The graph of a function is shown in the figure. Make a rough sketch of an antiderivative F, given that $F(0) = 1$.



46. The graph of the velocity function of a particle is shown in the figure. Sketch the graph of a position function.



47. The graph of f' is shown in the figure. Sketch the graph of f if f is continuous and $f(0) = -1$.



48. (a) Use a graphing device to graph $f(x) = 2x - 3\sqrt{x}$.
(b) Starting with the graph in part (a), sketch a rough graph of the antiderivative F that satisfies $F(0) = 1$.
(c) Use the rules of this section to find an expression for $F(x)$.
(d) Graph F using the expression in part (c). Compare with your sketch in part (b).

49–50 Draw a graph of f and use it to make a rough sketch of the antiderivative that passes through the origin.

49. $f(x) = \dfrac{\sin x}{1 + x^2}, \quad -2\pi \leqslant x \leqslant 2\pi$

50. $f(x) = \sqrt{x^4 - 2x^2 + 2} - 2, \quad -3 \leqslant x \leqslant 3$

51–56 A particle is moving with the given data. Find the position of the particle.

51. $v(t) = \sin t - \cos t, \quad s(0) = 0$

52. $v(t) = 1.5\sqrt{t}, \quad s(4) = 10$

53. $a(t) = 2t + 1, \quad s(0) = 3, \quad v(0) = -2$

54. $a(t) = 3\cos t - 2\sin t, \quad s(0) = 0, \quad v(0) = 4$

55. $a(t) = 10\sin t + 3\cos t, \quad s(0) = 0, \quad s(2\pi) = 12$

56. $a(t) = t^2 - 4t + 6, \quad s(0) = 0, \quad s(1) = 20$

57. A stone is dropped from the upper observation deck (the Space Deck) of the CN Tower, 450 m above the ground.
(a) Find the distance of the stone above ground level at time t.
(b) How long does it take the stone to reach the ground?
(c) With what velocity does it strike the ground?
(d) If the stone is thrown downward with a speed of 5 m/s, how long does it take to reach the ground?

58. Show that for motion in a straight line with constant acceleration a, initial velocity v_0, and initial displacement s_0, the displacement after time t is

$$s = \tfrac{1}{2}at^2 + v_0 t + s_0$$

59. An object is projected upward with initial velocity v_0 meters per second from a point s_0 meters above the ground. Show that

$$[v(t)]^2 = v_0^2 - 19.6[s(t) - s_0]$$

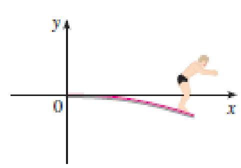
60. Two balls are thrown upward from the edge of the cliff in Example 7. The first is thrown with a speed of 48 ft/s and the other is thrown a second later with a speed of 24 ft/s. Do the balls ever pass each other?

61. A stone was dropped off a cliff and hit the ground with a speed of 120 ft/s. What is the height of the cliff?

62. If a diver of mass m stands at the end of a diving board with length L and linear density ρ, then the board takes on the shape of a curve $y = f(x)$, where

$$EIy'' = mg(L - x) + \tfrac{1}{2}\rho g(L - x)^2$$

E and I are positive constants that depend on the material of the board and g (< 0) is the acceleration due to gravity.
(a) Find an expression for the shape of the curve.
(b) Use $f(L)$ to estimate the distance below the horizontal at the end of the board.



63. A company estimates that the marginal cost (in dollars per item) of producing x items is $1.92 - 0.002x$. If the cost of producing one item is \$562, find the cost of producing 100 items.

64. The linear density of a rod of length 1 m is given by $\rho(x) = 1/\sqrt{x}$, in grams per centimeter, where x is measured in centimeters from one end of the rod. Find the mass of the rod.

65. Since raindrops grow as they fall, their surface area increases and therefore the resistance to their falling increases. A raindrop has an initial downward velocity of 10 m/s and its downward acceleration is

$$a = \begin{cases} 9 - 0.9t & \text{if } 0 \leq t \leq 10 \\ 0 & \text{if } t > 10 \end{cases}$$

If the raindrop is initially 500 m above the ground, how long does it take to fall?

66. A car is traveling at 50 mi/h when the brakes are fully applied, producing a constant deceleration of 22 ft/s^2. What is the distance traveled before the car comes to a stop?

67. What constant acceleration is required to increase the speed of a car from 30 mi/h to 50 mi/h in 5 s?

68. A car braked with a constant deceleration of 16 ft/s^2, producing skid marks measuring 200 ft before coming to a stop. How fast was the car traveling when the brakes were first applied?

69. A car is traveling at 100 km/h when the driver sees an accident 80 m ahead and slams on the brakes. What constant deceleration is required to stop the car in time to avoid a pileup?

70. A model rocket is fired vertically upward from rest. Its acceleration for the first three seconds is $a(t) = 60t$, at which time the fuel is exhausted and it becomes a freely "falling" body. Fourteen seconds later, the rocket's parachute opens, and the (downward) velocity slows linearly to -18 ft/s in 5 s. The rocket then "floats" to the ground at that rate.

(a) Determine the position function s and the velocity function v (for all times t). Sketch the graphs of s and v.
(b) At what time does the rocket reach its maximum height, and what is that height?
(c) At what time does the rocket land?

71. A high-speed bullet train accelerates and decelerates at the rate of 4 ft/s^2. Its maximum cruising speed is 90 mi/h.

(a) What is the maximum distance the train can travel if it accelerates from rest until it reaches its cruising speed and then runs at that speed for 15 minutes?
(b) Suppose that the train starts from rest and must come to a complete stop in 15 minutes. What is the maximum distance it can travel under these conditions?
(c) Find the minimum time that the train takes to travel between two consecutive stations that are 45 miles apart.
(d) The trip from one station to the next takes 37.5 minutes. How far apart are the stations?

3 Review

Concept Check

1. Explain the difference between an absolute maximum and a local maximum. Illustrate with a sketch.

2. (a) What does the Extreme Value Theorem say?
(b) Explain how the Closed Interval Method works.

3. (a) State Fermat's Theorem.
(b) Define a critical number of f.

4. (a) State Rolle's Theorem.
(b) State the Mean Value Theorem and give a geometric interpretation.

5. (a) State the Increasing/Decreasing Test.
(b) What does it mean to say that f is concave upward on an interval I?
(c) State the Concavity Test.
(d) What are inflection points? How do you find them?

6. (a) State the First Derivative Test.
(b) State the Second Derivative Test.
(c) What are the relative advantages and disadvantages of these tests?

7. Explain the meaning of each of the following statements.
(a) $\lim_{x\to\infty} f(x) = L$ (b) $\lim_{x\to-\infty} f(x) = L$ (c) $\lim_{x\to\infty} f(x) = \infty$
(d) The curve $y = f(x)$ has the horizontal asymptote $y = L$.

8. If you have a graphing calculator or computer, why do you need calculus to graph a function?

9. (a) Given an initial approximation x_1 to a root of the equation $f(x) = 0$, explain geometrically, with a diagram, how the second approximation x_2 in Newton's method is obtained.
(b) Write an expression for x_2 in terms of x_1, $f(x_1)$, and $f'(x_1)$.
(c) Write an expression for x_{n+1} in terms of x_n, $f(x_n)$, and $f'(x_n)$.
(d) Under what circumstances is Newton's method likely to fail or to work very slowly?

10. (a) What is an antiderivative of a function f?
(b) Suppose F_1 and F_2 are both antiderivatives of f on an interval I. How are F_1 and F_2 related?

True-False Quiz

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. If $f'(c) = 0$, then f has a local maximum or minimum at c.
2. If f has an absolute minimum value at c, then $f'(c) = 0$.
3. If f is continuous on (a, b), then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in (a, b).
4. If f is differentiable and $f(-1) = f(1)$, then there is a number c such that $|c| < 1$ and $f'(c) = 0$.
5. If $f'(x) < 0$ for $1 < x < 6$, then f is decreasing on $(1, 6)$.
6. If $f''(2) = 0$, then $(2, f(2))$ is an inflection point of the curve $y = f(x)$.
7. If $f'(x) = g'(x)$ for $0 < x < 1$, then $f(x) = g(x)$ for $0 < x < 1$.
8. There exists a function f such that $f(1) = -2$, $f(3) = 0$, and $f'(x) > 1$ for all x.
9. There exists a function f such that $f(x) > 0$, $f'(x) < 0$, and $f''(x) > 0$ for all x.
10. There exists a function f such that $f(x) < 0$, $f'(x) < 0$, and $f''(x) > 0$ for all x.
11. If f and g are increasing on an interval I, then $f + g$ is increasing on I.
12. If f and g are increasing on an interval I, then $f - g$ is increasing on I.
13. If f and g are increasing on an interval I, then fg is increasing on I.
14. If f and g are positive increasing functions on an interval I, then fg is increasing on I.
15. If f is increasing and $f(x) > 0$ on I, then $g(x) = 1/f(x)$ is decreasing on I.
16. If f is even, then f' is even.
17. If f is periodic, then f' is periodic.
18. The most general antiderivative of $f(x) = x^{-2}$ is
$$F(x) = -\frac{1}{x} + C$$
19. If $f'(x)$ exists and is nonzero for all x, then $f(1) \neq f(0)$.

Exercises

1–6 Find the local and absolute extreme values of the function on the given interval.

1. $f(x) = x^3 - 6x^2 + 9x + 1$, $[2, 4]$
2. $f(x) = x\sqrt{1 - x}$, $[-1, 1]$
3. $f(x) = \dfrac{3x - 4}{x^2 + 1}$, $[-2, 2]$
4. $f(x) = \sqrt{x^2 + x + 1}$, $[-2, 1]$
5. $f(x) = x + 2\cos x$, $[-\pi, \pi]$
6. $f(x) = \sin x + \cos^2 x$, $[0, \pi]$

7–12 Find the limit.

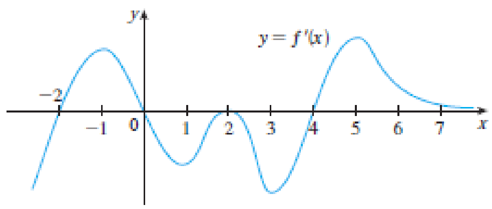
7. $\displaystyle\lim_{x\to\infty} \frac{3x^4 + x - 5}{6x^4 - 2x^2 + 1}$
8. $\displaystyle\lim_{t\to\infty} \frac{t^3 - t + 2}{(2t - 1)(t^2 + t + 1)}$
9. $\displaystyle\lim_{x\to-\infty} \frac{\sqrt{4x^2 + 1}}{3x - 1}$
10. $\displaystyle\lim_{x\to-\infty} (x^2 + x^3)$
11. $\displaystyle\lim_{x\to\infty} \left(\sqrt{4x^2 + 3x} - 2x\right)$
12. $\displaystyle\lim_{x\to\infty} \frac{\sin^4 x}{\sqrt{x}}$

13–15 Sketch the graph of a function that satisfies the given conditions.

13. $f(0) = 0$, $f'(-2) = f'(1) = f'(9) = 0$,
$\lim_{x\to\infty} f(x) = 0$, $\lim_{x\to 6} f(x) = -\infty$,
$f'(x) < 0$ on $(-\infty, -2)$, $(1, 6)$, and $(9, \infty)$,
$f'(x) > 0$ on $(-2, 1)$ and $(6, 9)$,
$f''(x) > 0$ on $(-\infty, 0)$ and $(12, \infty)$,
$f''(x) < 0$ on $(0, 6)$ and $(6, 12)$
14. $f(0) = 0$, f is continuous and even,
$f'(x) = 2x$ if $0 < x < 1$, $f'(x) = -1$ if $1 < x < 3$,
$f'(x) = 1$ if $x > 3$
15. f is odd, $f'(x) < 0$ for $0 < x < 2$,
$f'(x) > 0$ for $x > 2$, $f''(x) > 0$ for $0 < x < 3$,
$f''(x) < 0$ for $x > 3$, $\lim_{x\to\infty} f(x) = -2$

16. The figure shows the graph of the *derivative* f' of a function f.
(a) On what intervals is f increasing or decreasing?
(b) For what values of x does f have a local maximum or minimum?

(c) Sketch the graph of f''.
(d) Sketch a possible graph of f.



17–28 Use the guidelines of Section 3.5 to sketch the curve.

17. $y = 2 - 2x - x^3$

18. $y = x^3 - 6x^2 - 15x + 4$

19. $y = x^4 - 3x^3 + 3x^2 - x$

20. $y = \dfrac{x}{1 - x^2}$

21. $y = \dfrac{1}{x(x - 3)^2}$

22. $y = \dfrac{1}{x^2} - \dfrac{1}{(x - 2)^2}$

23. $y = x^2/(x + 8)$

24. $y = \sqrt{1 - x} + \sqrt{1 + x}$

25. $y = x\sqrt{2 + x}$

26. $y = \sqrt[3]{x^2 + 1}$

27. $y = \sin^2 x - 2 \cos x$

28. $y = 4x - \tan x, \quad -\pi/2 < x < \pi/2$

29–32 Produce graphs of f that reveal all the important aspects of the curve. Use graphs of f' and f'' to estimate the intervals of increase and decrease, extreme values, intervals of concavity, and inflection points. In Exercise 29 use calculus to find these quantities exactly.

29. $f(x) = \dfrac{x^2 - 1}{x^3}$

30. $f(x) = \dfrac{x^3 - x}{x^2 + x + 3}$

31. $f(x) = 3x^6 - 5x^5 + x^4 - 5x^3 - 2x^2 + 2$

32. $f(x) = x^2 + 6.5 \sin x, \quad -5 \leqslant x \leqslant 5$

33. Show that the equation $3x + 2 \cos x + 5 = 0$ has exactly one real root.

34. Suppose that f is continuous on $[0, 4]$, $f(0) = 1$, and $2 \leqslant f'(x) \leqslant 5$ for all x in $(0, 4)$. Show that $9 \leqslant f(4) \leqslant 21$.

35. By applying the Mean Value Theorem to the function $f(x) = x^{1/5}$ on the interval $[32, 33]$, show that

$$2 < \sqrt[5]{33} < 2.0125$$

36. For what values of the constants a and b is $(1, 3)$ a point of inflection of the curve $y = ax^3 + bx^2$?

37. Let $g(x) = f(x^2)$, where f is twice differentiable for all x, $f'(x) > 0$ for all $x \neq 0$, and f is concave downward on $(-\infty, 0)$ and concave upward on $(0, \infty)$.
(a) At what numbers does g have an extreme value?
(b) Discuss the concavity of g.

38. Find two positive integers such that the sum of the first number and four times the second number is 1000 and the product of the numbers is as large as possible.

39. Show that the shortest distance from the point (x_1, y_1) to the straight line $Ax + By + C = 0$ is

$$\frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$

40. Find the point on the hyperbola $xy = 8$ that is closest to the point $(3, 0)$.

41. Find the smallest possible area of an isosceles triangle that is circumscribed about a circle of radius r.

42. Find the volume of the largest circular cone that can be inscribed in a sphere of radius r.

43. In ΔABC, D lies on AB, $CD \perp AB$, $|AD| = |BD| = 4$ cm, and $|CD| = 5$ cm. Where should a point P be chosen on CD so that the sum $|PA| + |PB| + |PC|$ is a minimum?

44. Solve Exercise 43 when $|CD| = 2$ cm.

45. The velocity of a wave of length L in deep water is

$$v = K\sqrt{\frac{L}{C} + \frac{C}{L}}$$

where K and C are known positive constants. What is the length of the wave that gives the minimum velocity?

46. A metal storage tank with volume V is to be constructed in the shape of a right circular cylinder surmounted by a hemisphere. What dimensions will require the least amount of metal?

47. A hockey team plays in an arena with a seating capacity of 15,000 spectators. With the ticket price set at \$12, average attendance at a game has been 11,000. A market survey indicates that for each dollar the ticket price is lowered, average attendance will increase by 1000. How should the owners of the team set the ticket price to maximize their revenue from ticket sales?

48. A manufacturer determines that the cost of making x units of a commodity is $C(x) = 1800 + 25x - 0.2x^2 + 0.001x^3$ and the demand function is $p(x) = 48.2 - 0.03x$.
(a) Graph the cost and revenue functions and use the graphs to estimate the production level for maximum profit.
(b) Use calculus to find the production level for maximum profit.
(c) Estimate the production level that minimizes the average cost.

49. Use Newton's method to find the root of the equation $x^5 - x^4 + 3x^2 - 3x - 2 = 0$ in the interval $[1, 2]$ correct to six decimal places.

50. Use Newton's method to find all roots of the equation $\sin x = x^2 - 3x + 1$ correct to six decimal places.

51. Use Newton's method to find the absolute maximum value of the function $f(t) = \cos t + t - t^2$ correct to eight decimal places.

52. Use the guidelines in Section 3.5 to sketch the curve $y = x \sin x$, $0 \leq x \leq 2\pi$. Use Newton's method when necessary.

53–58 Find f.

53. $f'(x) = \sqrt{x^3} + \sqrt[3]{x^2}$

54. $f'(x) = 8x - 3\sec^2 x$

55. $f'(t) = 2t - 3\sin t$, $\quad f(0) = 5$

56. $f'(u) = \dfrac{u^2 + \sqrt{u}}{u}$, $\quad f(1) = 3$

57. $f''(x) = 1 - 6x + 48x^2$, $\quad f(0) = 1$, $\quad f'(0) = 2$

58. $f''(x) = 2x^3 + 3x^2 - 4x + 5$, $\quad f(0) = 2$, $\quad f(1) = 0$

59–60 A particle is moving with the given data. Find the position of the particle.

59. $v(t) = 2t - \sin t$, $\quad s(0) = 3$

60. $a(t) = \sin t + 3\cos t$, $\quad s(0) = 0$, $\quad v(0) = 2$

61. Use a graphing device to draw a graph of the function $f(x) = x^2 \sin(x^2)$, $0 \leq x \leq \pi$, and use that graph to sketch the antiderivative F of f that satisfies the initial condition $F(0) = 0$.

62. Investigate the family of curves given by

$$f(x) = x^4 + x^3 + cx^2$$

In particular you should determine the transitional value of c at which the number of critical numbers changes and the transitional value at which the number of inflection points changes. Illustrate the various possible shapes with graphs.

63. A canister is dropped from a helicopter 500 m above the ground. Its parachute does not open, but the canister has been designed to withstand an impact velocity of 100 m/s. Will it burst?

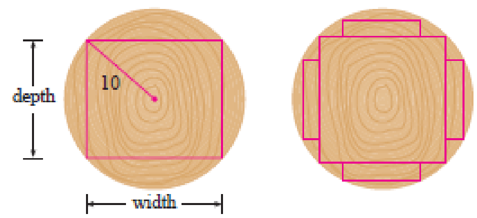
64. In an automobile race along a straight road, car A passed car B twice. Prove that at some time during the race their accelerations were equal. State the assumptions that you make.

65. A rectangular beam will be cut from a cylindrical log of radius 10 inches.

(a) Show that the beam of maximal cross-sectional area is a square.

(b) Four rectangular planks will be cut from the four sections of the log that remain after cutting the square beam. Determine the dimensions of the planks that will have maximal cross-sectional area.

(c) Suppose that the strength of a rectangular beam is proportional to the product of its width and the square of its depth. Find the dimensions of the strongest beam that can be cut from the cylindrical log.



66. If a projectile is fired with an initial velocity v at an angle of inclination θ from the horizontal, then its trajectory, neglecting air resistance, is the parabola

$$y = (\tan\theta)x - \frac{g}{2v^2\cos^2\theta}x^2 \qquad 0 < \theta < \frac{\pi}{2}$$

(a) Suppose the projectile is fired from the base of a plane that is inclined at an angle α, $\alpha > 0$, from the horizontal, as shown in the figure. Show that the range of the projectile, measured up the slope, is given by

$$R(\theta) = \frac{2v^2\cos\theta\,\sin(\theta - \alpha)}{g\cos^2\alpha}$$

(b) Determine θ so that R is a maximum.

(c) Suppose the plane is at an angle α *below* the horizontal. Determine the range R in this case, and determine the angle at which the projectile should be fired to maximize R.