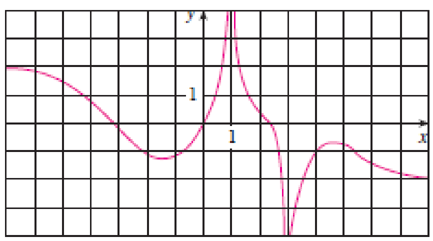
3.4 Exercises

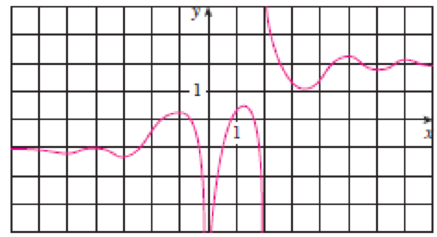
1. Explain in your own words the meaning of each of the following.
   (a) $\lim_{x \to \infty} f(x) = 5$ (b) $\lim_{x \to -\infty} f(x) = 3$

2. (a) Can the graph of $y = f(x)$ intersect a vertical asymptote? Can it intersect a horizontal asymptote? Illustrate by sketching graphs.
   (b) How many horizontal asymptotes can the graph of $y = f(x)$ have? Sketch graphs to illustrate the possibilities.

3. For the function f whose graph is given, state the following.
   (a) $\lim_{x \to \infty} f(x)$ (b) $\lim_{x \to -\infty} f(x)$
   (c) $\lim_{x \to 1} f(x)$ (d) $\lim_{x \to 3} f(x)$
   (e) The equations of the asymptotes



4. For the function g whose graph is given, state the following.
   (a) $\lim_{x \to \infty} g(x)$ (b) $\lim_{x \to -\infty} g(x)$
   (c) $\lim_{x \to 0} g(x)$ (d) $\lim_{x \to 2^-} g(x)$
   (e) $\lim_{x \to 2^+} g(x)$ (f) The equations of the asymptotes



5. Guess the value of the limit
$$\lim_{x \to \infty} \frac{x^2}{2^x}$$
by evaluating the function $f(x) = x^2/2^x$ for $x = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 20, 50$, and 100. Then use a graph of f to support your guess.

6. (a) Use a graph of
$$f(x) = \left(1 - \frac{2}{x}\right)^x$$
to estimate the value of $\lim_{x \to \infty} f(x)$ correct to two decimal places.
   (b) Use a table of values of $f(x)$ to estimate the limit to four decimal places.

7–8 Evaluate the limit and justify each step by indicating the appropriate properties of limits.

7. $\lim_{x \to \infty} \dfrac{3x^2 - x + 4}{2x^2 + 5x - 8}$

8. $\lim_{x \to \infty} \sqrt{\dfrac{12x^3 - 5x + 2}{1 + 4x^2 + 3x^3}}$

9–30 Find the limit or show that it does not exist.

9. $\lim_{x \to \infty} \dfrac{3x - 2}{2x + 1}$

10. $\lim_{x \to \infty} \dfrac{1 - x^2}{x^3 - x + 1}$

11. $\lim_{x \to -\infty} \dfrac{x - 2}{x^2 + 1}$

12. $\lim_{x \to -\infty} \dfrac{4x^3 + 6x^2 - 2}{2x^3 - 4x + 5}$

13. $\lim_{t \to \infty} \dfrac{\sqrt{t} + t^2}{2t - t^2}$

14. $\lim_{t \to \infty} \dfrac{t - t\sqrt{t}}{2t^{3/2} + 3t - 5}$

15. $\lim_{x \to \infty} \dfrac{(2x^2 + 1)^2}{(x - 1)^2(x^2 + x)}$

16. $\lim_{x \to \infty} \dfrac{x^2}{\sqrt{x^4 + 1}}$

17. $\lim_{x \to \infty} \dfrac{\sqrt{9x^6 - x}}{x^3 + 1}$

18. $\lim_{x \to -\infty} \dfrac{\sqrt{9x^6 - x}}{x^3 + 1}$

19. $\lim_{x \to \infty} \left(\sqrt{9x^2 + x} - 3x\right)$

20. $\lim_{x \to -\infty} \left(x + \sqrt{x^2 + 2x}\right)$

21. $\lim_{x \to \infty} \left(\sqrt{x^2 + ax} - \sqrt{x^2 + bx}\right)$

22. $\lim_{x \to \infty} \cos x$

23. $\lim_{x \to \infty} \dfrac{x^4 - 3x^2 + x}{x^3 - x + 2}$

24. $\lim_{x \to \infty} \sqrt{x^2 + 1}$

25. $\lim_{x \to -\infty} (x^4 + x^5)$

26. $\lim_{x \to -\infty} \dfrac{1 + x^6}{x^4 + 1}$

27. $\lim_{x \to \infty} \left(x - \sqrt{x}\right)$

28. $\lim_{x \to \infty} (x^2 - x^4)$

29. $\lim_{x \to \infty} x \sin \dfrac{1}{x}$

30. $\lim_{x \to \infty} \sqrt{x} \sin \dfrac{1}{x}$

31. (a) Estimate the value of

$$\lim_{x \to -\infty} \left(\sqrt{x^2 + x + 1} + x\right)$$

by graphing the function $f(x) = \sqrt{x^2 + x + 1} + x$.

(b) Use a table of values of $f(x)$ to guess the value of the limit.

(c) Prove that your guess is correct.

32. (a) Use a graph of

$$f(x) = \sqrt{3x^2 + 8x + 6} - \sqrt{3x^2 + 3x + 1}$$

to estimate the value of $\lim_{x \to \infty} f(x)$ to one decimal place.

(b) Use a table of values of $f(x)$ to estimate the limit to four decimal places.

(c) Find the exact value of the limit.

33–38 Find the horizontal and vertical asymptotes of each curve. If you have a graphing device, check your work by graphing the curve and estimating the asymptotes.

33. $y = \dfrac{2x + 1}{x - 2}$

34. $y = \dfrac{x^2 + 1}{2x^2 - 3x - 2}$

35. $y = \dfrac{2x^2 + x - 1}{x^2 + x - 2}$

36. $y = \dfrac{1 + x^4}{x^2 - x^4}$

37. $y = \dfrac{x^3 - x}{x^2 - 6x + 5}$

38. $F(x) = \dfrac{x - 9}{\sqrt{4x^2 + 3x + 2}}$

39. Estimate the horizontal asymptote of the function

$$f(x) = \frac{3x^3 + 500x^2}{x^3 + 500x^2 + 100x + 2000}$$

by graphing f for $-10 \leqslant x \leqslant 10$. Then calculate the equation of the asymptote by evaluating the limit. How do you explain the discrepancy?

40. (a) Graph the function

$$f(x) = \frac{\sqrt{2x^2 + 1}}{3x - 5}$$

How many horizontal and vertical asymptotes do you observe? Use the graph to estimate the values of the limits

$$\lim_{x \to \infty} \frac{\sqrt{2x^2 + 1}}{3x - 5} \quad \text{and} \quad \lim_{x \to -\infty} \frac{\sqrt{2x^2 + 1}}{3x - 5}$$

(b) By calculating values of $f(x)$, give numerical estimates of the limits in part (a).

(c) Calculate the exact values of the limits in part (a). Did you get the same value or different values for these two limits? [In view of your answer to part (a), you might have to check your calculation for the second limit.]

41. Find a formula for a function f that satisfies the following conditions:

$\lim_{x \to \pm\infty} f(x) = 0, \quad \lim_{x \to 0} f(x) = -\infty, \quad f(2) = 0,$

$\lim_{x \to 3^-} f(x) = \infty, \quad \lim_{x \to 3^+} f(x) = -\infty$

42. Find a formula for a function that has vertical asymptotes $x = 1$ and $x = 3$ and horizontal asymptote $y = 1$.

43. A function f is a ratio of quadratic functions and has a vertical asymptote $x = 4$ and just one x-intercept, $x = 1$. It is known that f has a removable discontinuity at $x = -1$ and $\lim_{x \to -1} f(x) = 2$. Evaluate

(a) $f(0)$

(b) $\lim_{x \to \infty} f(x)$

44–47 Find the horizontal asymptotes of the curve and use them, together with concavity and intervals of increase and decrease, to sketch the curve.

44. $y = \dfrac{1 + 2x^2}{1 + x^2}$

45. $y = \dfrac{1 - x}{1 + x}$

46. $y = \dfrac{x}{\sqrt{x^2 + 1}}$

47. $y = \dfrac{x}{x^2 + 1}$

48–52 Find the limits as $x \to \infty$ and as $x \to -\infty$. Use this information, together with intercepts, to give a rough sketch of the graph as in Example 11.

48. $y = 2x^3 - x^4$

49. $y = x^4 - x^6$

50. $y = x^3(x + 2)^2(x - 1)$

51. $y = (3 - x)(1 + x)^2(1 - x)^4$

52. $y = x^2(x^2 - 1)^2(x + 2)$

53–56 Sketch the graph of a function that satisfies all of the given conditions.

53. $f'(2) = 0$, $f(2) = -1$, $f(0) = 0$,
$f'(x) < 0$ if $0 < x < 2$, $f'(x) > 0$ if $x > 2$,
$f''(x) < 0$ if $0 \leqslant x < 1$ or if $x > 4$,
$f''(x) > 0$ if $1 < x < 4$, $\lim_{x\to\infty} f(x) = 1$,
$f(-x) = f(x)$ for all x

54. $f'(2) = 0$, $f(0) = 1$, $f'(x) > 0$ if $0 < x < 2$,
$f'(x) < 0$ if $x > 2$, $f''(x) < 0$ if $0 < x < 4$,
$f''(x) > 0$ if $x > 4$, $\lim_{x\to\infty} f(x) = 0$,
$f(-x) = -f(x)$ for all x

55. $f(1) = f'(1) = 0$, $\lim_{x\to 2^+} f(x) = \infty$, $\lim_{x\to 2^-} f(x) = -\infty$,
$\lim_{x\to 0} f(x) = -\infty$, $\lim_{x\to -\infty} f(x) = \infty$, $\lim_{x\to\infty} f(x) = 0$,
$f''(x) > 0$ for $x > 2$, $f''(x) < 0$ for $x < 0$ and for $0 < x < 2$

56. $g(0) = 0$, $g''(x) < 0$ for $x \neq 0$, $\lim_{x\to -\infty} g(x) = \infty$,
$\lim_{x\to\infty} g(x) = -\infty$, $\lim_{x\to 0^-} g'(x) = -\infty$,
$\lim_{x\to 0^+} g'(x) = \infty$

57. (a) Use the Squeeze Theorem to evaluate $\displaystyle\lim_{x\to\infty} \frac{\sin x}{x}$.

(b) Graph $f(x) = (\sin x)/x$. How many times does the graph cross the asymptote?

58. By the *end behavior* of a function we mean the behavior of its values as $x \to \infty$ and as $x \to -\infty$.

(a) Describe and compare the end behavior of the functions

$$P(x) = 3x^5 - 5x^3 + 2x \qquad Q(x) = 3x^5$$

by graphing both functions in the viewing rectangles $[-2, 2]$ by $[-2, 2]$ and $[-10, 10]$ by $[-10{,}000, 10{,}000]$.

(b) Two functions are said to have the *same end behavior* if their ratio approaches 1 as $x \to \infty$. Show that P and Q have the same end behavior.

59. Let P and Q be polynomials. Find

$$\lim_{x\to\infty} \frac{P(x)}{Q(x)}$$

if the degree of P is (a) less than the degree of Q and (b) greater than the degree of Q.

60. Make a rough sketch of the curve $y = x^n$ (n an integer) for the following five cases:

(i) $n = 0$ (ii) $n > 0$, n odd
(iii) $n > 0$, n even (iv) $n < 0$, n odd
(v) $n < 0$, n even

Then use these sketches to find the following limits.

(a) $\displaystyle\lim_{x\to 0^+} x^n$ (b) $\displaystyle\lim_{x\to 0^-} x^n$

(c) $\displaystyle\lim_{x\to\infty} x^n$ (d) $\displaystyle\lim_{x\to -\infty} x^n$

61. Find $\lim_{x\to\infty} f(x)$ if

$$\frac{4x - 1}{x} < f(x) < \frac{4x^2 + 3x}{x^2}$$

for all $x > 5$.

62. (a) A tank contains 5000 L of pure water. Brine that contains 30 g of salt per liter of water is pumped into the tank at a rate of 25 L/min. Show that the concentration of salt after t minutes (in grams per liter) is

$$C(t) = \frac{30t}{200 + t}$$

(b) What happens to the concentration as $t \to \infty$?

63. Use a graph to find a number N such that

$$\text{if} \quad x > N \quad \text{then} \quad \left| \frac{3x^2 + 1}{2x^2 + x + 1} - 1.5 \right| < 0.05$$

64. For the limit

$$\lim_{x\to\infty} \frac{\sqrt{4x^2 + 1}}{x + 1} = 2$$

illustrate Definition 5 by finding values of N that correspond to $\varepsilon = 0.5$ and $\varepsilon = 0.1$.

65. For the limit

$$\lim_{x\to -\infty} \frac{\sqrt{4x^2 + 1}}{x + 1} = -2$$

illustrate Definition 6 by finding values of N that correspond to $\varepsilon = 0.5$ and $\varepsilon = 0.1$.

66. For the limit

$$\lim_{x\to\infty} \frac{2x + 1}{\sqrt{x + 1}} = \infty$$

illustrate Definition 7 by finding a value of N that corresponds to $M = 100$.

67. (a) How large do we have to take x so that $1/x^2 < 0.0001$?

(b) Taking $r = 2$ in Theorem 4, we have the statement

$$\lim_{x\to\infty} \frac{1}{x^2} = 0$$

Prove this directly using Definition 5.

68. (a) How large do we have to take x so that $1/\sqrt{x} < 0.0001$?
(b) Taking $r = \frac{1}{2}$ in Theorem 4, we have the statement

$$\lim_{x \to \infty} \frac{1}{\sqrt{x}} = 0$$

Prove this directly using Definition 5.

69. Use Definition 6 to prove that $\lim_{x \to -\infty} \frac{1}{x} = 0$.

70. Prove, using Definition 7, that $\lim_{x \to \infty} x^3 = \infty$.

71. Prove that

$$\lim_{x \to \infty} f(x) = \lim_{t \to 0^+} f(1/t)$$

and $$\lim_{x \to -\infty} f(x) = \lim_{t \to 0^-} f(1/t)$$

if these limits exist.

72. Formulate a precise definition of

$$\lim_{x \to -\infty} f(x) = -\infty$$

Then use your definition to prove that

$$\lim_{x \to -\infty} (1 + x^3) = -\infty$$