2.9 Exercises

1–4 Find the linearization $L(x)$ of the function at a.

1. $f(x) = x^4 + 3x^2, \quad a = -1$

2. $f(x) = \sin x, \quad a = \pi/6$

3. $f(x) = \sqrt{x}, \quad a = 4$

4. $f(x) = x^{3/4}, \quad a = 16$

5. Find the linear approximation of the function $f(x) = \sqrt{1 - x}$ at $a = 0$ and use it to approximate the numbers $\sqrt{0.9}$ and $\sqrt{0.99}$. Illustrate by graphing f and the tangent line.

6. Find the linear approximation of the function $g(x) = \sqrt[3]{1 + x}$ at $a = 0$ and use it to approximate the numbers $\sqrt[3]{0.95}$ and $\sqrt[3]{1.1}$. Illustrate by graphing g and the tangent line.

7–10 Verify the given linear approximation at $a = 0$. Then determine the values of x for which the linear approximation is accurate to within 0.1.

7. $\sqrt[4]{1 + 2x} \approx 1 + \frac{1}{2}x$

8. $(1 + x)^{-3} \approx 1 - 3x$

9. $1/(1 + 2x)^4 \approx 1 - 8x$

10. $\tan x \approx x$

11–14 Find the differential of each function.

11. (a) $y = x^2 \sin 2x$ (b) $y = \sqrt{1 + t^2}$

12. (a) $y = s/(1 + 2s)$ (b) $y = u \cos u$

13. (a) $y = \tan \sqrt{t}$ (b) $y = \dfrac{1 - v^2}{1 + v^2}$

14. (a) $y = (t + \tan t)^5$ (b) $y = \sqrt{z + 1/z}$

15–18 (a) Find the differential dy and (b) evaluate dy for the given values of x and dx.

15. $y = \tan x, \quad x = \pi/4, \quad dx = -0.1$

16. $y = \cos \pi x, \quad x = \frac{1}{3}, \quad dx = -0.02$

17. $y = \sqrt{3 + x^2}, \quad x = 1, \quad dx = -0.1$

18. $y = \dfrac{x + 1}{x - 1}, \quad x = 2, \quad dx = 0.05$

19–22 Compute Δy and dy for the given values of x and $dx = \Delta x$. Then sketch a diagram like Figure 5 showing the line segments with lengths dx, dy, and Δy.

19. $y = 2x - x^2$, $x = 2$, $\Delta x = -0.4$

20. $y = \sqrt{x}$, $x = 1$, $\Delta x = 1$

21. $y = 2/x$, $x = 4$, $\Delta x = 1$

22. $y = x^3$, $x = 1$, $\Delta x = 0.5$

23–28 Use a linear approximation (or differentials) to estimate the given number.

23. $(1.999)^4$ **24.** $\sin 1°$ **25.** $\sqrt[3]{1001}$

26. $1/4.002$ **27.** $\tan 44°$ **28.** $\sqrt{99.8}$

29–30 Explain, in terms of linear approximations or differentials, why the approximation is reasonable.

29. $\sec 0.08 \approx 1$ **30.** $(1.01)^6 \approx 1.06$

31. The edge of a cube was found to be 30 cm with a possible error in measurement of 0.1 cm. Use differentials to estimate the maximum possible error, relative error, and percentage error in computing (a) the volume of the cube and (b) the surface area of the cube.

32. The radius of a circular disk is given as 24 cm with a maximum error in measurement of 0.2 cm.
(a) Use differentials to estimate the maximum error in the calculated area of the disk.
(b) What is the relative error? What is the percentage error?

33. The circumference of a sphere was measured to be 84 cm with a possible error of 0.5 cm.
(a) Use differentials to estimate the maximum error in the calculated surface area. What is the relative error?
(b) Use differentials to estimate the maximum error in the calculated volume. What is the relative error?

34. Use differentials to estimate the amount of paint needed to apply a coat of paint 0.05 cm thick to a hemispherical dome with diameter 50 m.

35. (a) Use differentials to find a formula for the approximate volume of a thin cylindrical shell with height h, inner radius r, and thickness Δr.
(b) What is the error involved in using the formula from part (a)?

36. One side of a right triangle is known to be 20 cm long and the opposite angle is measured as 30°, with a possible error of $\pm 1°$.
(a) Use differentials to estimate the error in computing the length of the hypotenuse.
(b) What is the percentage error?

37. If a current I passes through a resistor with resistance R, Ohm's Law states that the voltage drop is $V = RI$. If V is constant and R is measured with a certain error, use differentials to show that the relative error in calculating I is approximately the same (in magnitude) as the relative error in R.

38. When blood flows along a blood vessel, the flux F (the volume of blood per unit time that flows past a given point) is proportional to the fourth power of the radius R of the blood vessel:

$$F = kR^4$$

(This is known as Poiseuille's Law; we will show why it is true in Section 8.4.) A partially clogged artery can be expanded by an operation called angioplasty, in which a balloon-tipped catheter is inflated inside the artery in order to widen it and restore the normal blood flow.

Show that the relative change in F is about four times the relative change in R. How will a 5% increase in the radius affect the flow of blood?

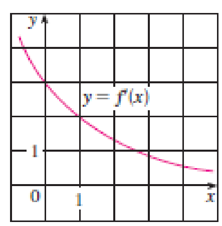
39. Establish the following rules for working with differentials (where c denotes a constant and u and v are functions of x).
(a) $dc = 0$ (b) $d(cu) = c\,du$
(c) $d(u + v) = du + dv$ (d) $d(uv) = u\,dv + v\,du$
(e) $d\left(\dfrac{u}{v}\right) = \dfrac{v\,du - u\,dv}{v^2}$ (f) $d(x^n) = nx^{n-1}\,dx$

40. On page 431 of *Physics: Calculus*, 2d ed., by Eugene Hecht (Pacific Grove, CA, 2000), in the course of deriving the formula $T = 2\pi\sqrt{L/g}$ for the period of a pendulum of length L, the author obtains the equation $a_T = -g\sin\theta$ for the tangential acceleration of the bob of the pendulum. He then says, "for small angles, the value of θ in radians is very nearly the value of $\sin\theta$; they differ by less than 2% out to about 20°."
(a) Verify the linear approximation at 0 for the sine function:

$$\sin x \approx x$$

(b) Use a graphing device to determine the values of x for which $\sin x$ and x differ by less than 2%. Then verify Hecht's statement by converting from radians to degrees.

41. Suppose that the only information we have about a function f is that $f(1) = 5$ and the graph of its *derivative* is as shown.
(a) Use a linear approximation to estimate $f(0.9)$ and $f(1.1)$.
(b) Are your estimates in part (a) too large or too small? Explain.



42. Suppose that we don't have a formula for $g(x)$ but we know that $g(2) = -4$ and $g'(x) = \sqrt{x^2 + 5}$ for all x.
(a) Use a linear approximation to estimate $g(1.95)$ and $g(2.05)$.
(b) Are your estimates in part (a) too large or too small? Explain.

2 Review

Concept Check

1. Write an expression for the slope of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$.

2. Suppose an object moves along a straight line with position $f(t)$ at time t. Write an expression for the instantaneous velocity of the object at time $t = a$. How can you interpret this velocity in terms of the graph of f?

3. If $y = f(x)$ and x changes from x_1 to x_2, write expressions for the following.
   (a) The average rate of change of y with respect to x over the interval $[x_1, x_2]$.
   (b) The instantaneous rate of change of y with respect to x at $x = x_1$.

4. Define the derivative $f'(a)$. Discuss two ways of interpreting this number.

5. (a) What does it mean for f to be differentiable at a?
   (b) What is the relation between the differentiability and continuity of a function?
   (c) Sketch the graph of a function that is continuous but not differentiable at $a = 2$.

6. Describe several ways in which a function can fail to be differentiable. Illustrate with sketches.

7. What are the second and third derivatives of a function f? If f is the position function of an object, how can you interpret f'' and f'''?

8. State each differentiation rule both in symbols and in words.
   (a) The Power Rule (b) The Constant Multiple Rule
   (c) The Sum Rule (d) The Difference Rule
   (e) The Product Rule (f) The Quotient Rule
   (g) The Chain Rule

9. State the derivative of each function.
   (a) $y = x^n$ (b) $y = \sin x$ (c) $y = \cos x$
   (d) $y = \tan x$ (e) $y = \csc x$ (f) $y = \sec x$
   (g) $y = \cot x$

10. Explain how implicit differentiation works.

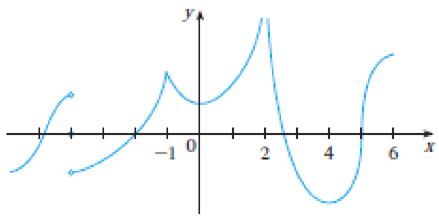
11. Give several examples of how the derivative can be interpreted as a rate of change in physics, chemistry, biology, economics, or other sciences.

12. (a) Write an expression for the linearization of f at a.
    (b) If $y = f(x)$, write an expression for the differential dy.
    (c) If $dx = \Delta x$, draw a picture showing the geometric meanings of Δy and dy.

Exercises

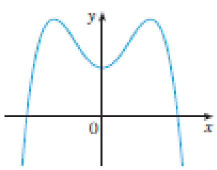
1. The displacement (in meters) of an object moving in a straight line is given by $s = 1 + 2t + \frac{1}{4}t^2$, where t is measured in seconds.
(a) Find the average velocity over each time period.
(i) $[1, 3]$ (ii) $[1, 2]$
(iii) $[1, 1.5]$ (iv) $[1, 1.1]$
(b) Find the instantaneous velocity when $t = 1$.

2. The graph of f is shown. State, with reasons, the numbers at which f is not differentiable.



3–4 Trace or copy the graph of the function. Then sketch a graph of its derivative directly beneath.

3.



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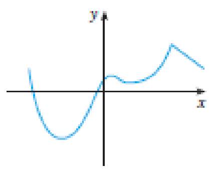
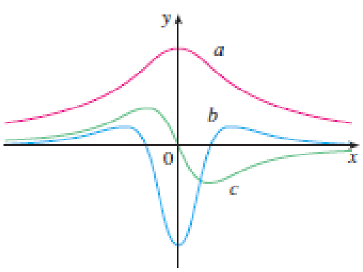


5. The figure shows the graphs of f, f', and f''. Identify each curve, and explain your choices.

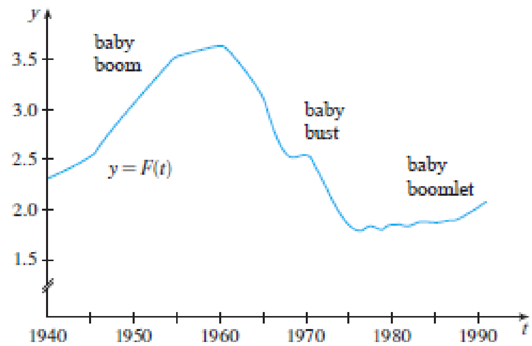


6. Find a function f and a number a such that

$$\lim_{h \to 0} \frac{(2 + h)^6 - 64}{h} = f'(a)$$

7. The total cost of repaying a student loan at an interest rate of r% per year is $C = f(r)$.
(a) What is the meaning of the derivative $f'(r)$? What are its units?
(b) What does the statement $f'(10) = 1200$ mean?
(c) Is $f'(r)$ always positive or does it change sign?

8. The *total fertility rate* at time t, denoted by $F(t)$, is an estimate of the average number of children born to each woman (assuming that current birth rates remain constant). The graph of the total fertility rate in the United States shows the fluctuations from 1940 to 1990.
(a) Estimate the values of $F'(1950)$, $F'(1965)$, and $F'(1987)$.
(b) What are the meanings of these derivatives?
(c) Can you suggest reasons for the values of these derivatives?



9. Let $C(t)$ be the total value of US currency (coins and banknotes) in circulation at time t. The table gives values of this function from 1980 to 2000, as of September 30, in billions of dollars. Interpret and estimate the value of $C'(1990)$.

t	1980	1985	1990	1995	2000
$C(t)$	129.9	187.3	271.9	409.3	568.6

10–11 Find $f'(x)$ from first principles, that is, directly from the definition of a derivative.

10. $f(x) = \dfrac{4 - x}{3 + x}$ **11.** $f(x) = x^3 + 5x + 4$

12. (a) If $f(x) = \sqrt{3 - 5x}$, use the definition of a derivative to find $f'(x)$.
(b) Find the domains of f and f'.
(c) Graph f and f' on a common screen. Compare the graphs to see whether your answer to part (a) is reasonable.

13–40 Calculate y'.

13. $y = (x^2 + x^3)^4$ **14.** $y = \dfrac{1}{\sqrt{x}} - \dfrac{1}{\sqrt[5]{x^3}}$

15. $y = \dfrac{x^2 - x + 2}{\sqrt{x}}$ **16.** $y = \dfrac{\tan x}{1 + \cos x}$

17. $y = x^2 \sin \pi x$ **18.** $y = \left(x + \dfrac{1}{x^2}\right)^{\sqrt{7}}$

19. $y = \dfrac{t^4 - 1}{t^4 + 1}$

20. $y = \sin(\cos x)$

21. $y = \tan\sqrt{1 - x}$

22. $y = \dfrac{1}{\sin(x - \sin x)}$

23. $xy^4 + x^2y = x + 3y$

24. $y = \sec(1 + x^2)$

25. $y = \dfrac{\sec 2\theta}{1 + \tan 2\theta}$

26. $x^2\cos y + \sin 2y = xy$

27. $y = (1 - x^{-1})^{-1}$

28. $y = 1/\sqrt[3]{x + \sqrt{x}}$

29. $\sin(xy) = x^2 - y$

30. $y = \sqrt{\sin\sqrt{x}}$

31. $y = \cot(3x^2 + 5)$

32. $y = \dfrac{(x + \lambda)^4}{x^4 + \lambda^4}$

33. $y = \sqrt{x}\cos\sqrt{x}$

34. $y = \dfrac{\sin mx}{x}$

35. $y = \tan^2(\sin\theta)$

36. $x\tan y = y - 1$

37. $y = \sqrt[5]{x\tan x}$

38. $y = \dfrac{(x - 1)(x - 4)}{(x - 2)(x - 3)}$

39. $y = \sin\left(\tan\sqrt{1 + x^3}\right)$

40. $y = \sin^2\left(\cos\sqrt{\sin\pi x}\right)$

41. If $f(t) = \sqrt{4t + 1}$, find $f''(2)$.

42. If $g(\theta) = \theta\sin\theta$, find $g''(\pi/6)$.

43. Find y'' if $x^6 + y^6 = 1$.

44. Find $f^{(n)}(x)$ if $f(x) = 1/(2 - x)$.

45–46 Find the limit.

45. $\displaystyle\lim_{x\to 0}\frac{\sec x}{1 - \sin x}$

46. $\displaystyle\lim_{t\to 0}\frac{t^3}{\tan^3 2t}$

47–48 Find an equation of the tangent to the curve at the given point.

47. $y = 4\sin^2 x$, $(\pi/6, 1)$

48. $y = \dfrac{x^2 - 1}{x^2 + 1}$, $(0, -1)$

49–50 Find equations of the tangent line and normal line to the curve at the given point.

49. $y = \sqrt{1 + 4\sin x}$, $(0, 1)$

50. $x^2 + 4xy + y^2 = 13$, $(2, 1)$

51. (a) If $f(x) = x\sqrt{5 - x}$, find $f'(x)$.
(b) Find equations of the tangent lines to the curve $y = x\sqrt{5 - x}$ at the points $(1, 2)$ and $(4, 4)$.
(c) Illustrate part (b) by graphing the curve and tangent lines on the same screen.
(d) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f'.

52. (a) If $f(x) = 4x - \tan x$, $-\pi/2 < x < \pi/2$, find f' and f''.
(b) Check to see that your answers to part (a) are reasonable by comparing the graphs of f, f', and f''.

53. At what points on the curve $y = \sin x + \cos x$, $0 \leqslant x \leqslant 2\pi$, is the tangent line horizontal?

54. Find the points on the ellipse $x^2 + 2y^2 = 1$ where the tangent line has slope 1.

55. Find a parabola $y = ax^2 + bx + c$ that passes through the point $(1, 4)$ and whose tangent lines at $x = -1$ and $x = 5$ have slopes 6 and -2, respectively.

56. How many tangent lines to the curve $y = x/(x + 1)$ pass through the point $(1, 2)$? At which points do these tangent lines touch the curve?

57. If $f(x) = (x - a)(x - b)(x - c)$, show that

$$\frac{f'(x)}{f(x)} = \frac{1}{x - a} + \frac{1}{x - b} + \frac{1}{x - c}$$

58. (a) By differentiating the double-angle formula

$$\cos 2x = \cos^2 x - \sin^2 x$$

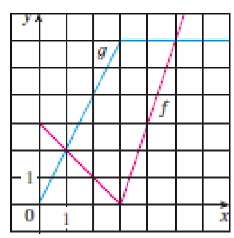
obtain the double-angle formula for the sine function.
(b) By differentiating the addition formula

$$\sin(x + a) = \sin x\cos a + \cos x\sin a$$

obtain the addition formula for the cosine function.

59. Suppose that $h(x) = f(x)g(x)$ and $F(x) = f(g(x))$, where $f(2) = 3$, $g(2) = 5$, $g'(2) = 4$, $f'(2) = -2$, and $f'(5) = 11$. Find (a) $h'(2)$ and (b) $F'(2)$.

60. If f and g are the functions whose graphs are shown, let $P(x) = f(x)g(x)$, $Q(x) = f(x)/g(x)$, and $C(x) = f(g(x))$. Find (a) $P'(2)$, (b) $Q'(2)$, and (c) $C'(2)$.



61–68 Find f' in terms of g'.

61. $f(x) = x^2g(x)$

62. $f(x) = g(x^2)$

63. $f(x) = [g(x)]^2$

64. $f(x) = x^a g(x^b)$

65. $f(x) = g(g(x))$

66. $f(x) = \sin(g(x))$

67. $f(x) = g(\sin x)$

68. $f(x) = g\left(\tan\sqrt{x}\right)$

69–71 Find h' in terms of f' and g'.

69. $h(x) = \dfrac{f(x)g(x)}{f(x) + g(x)}$

70. $h(x) = \sqrt{\dfrac{f(x)}{g(x)}}$

71. $h(x) = f(g(\sin 4x))$

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72. A particle moves along a horizontal line so that its coordinate at time t is $x = \sqrt{b^2 + c^2t^2}$, $t \geqslant 0$, where b and c are positive constants.
(a) Find the velocity and acceleration functions.
(b) Show that the particle always moves in the positive direction.

73. A particle moves on a vertical line so that its coordinate at time t is $y = t^3 - 12t + 3$, $t \geqslant 0$.
(a) Find the velocity and acceleration functions.
(b) When is the particle moving upward and when is it moving downward?
(c) Find the distance that the particle travels in the time interval $0 \leqslant t \leqslant 3$.
(d) Graph the position, velocity, and acceleration functions for $0 \leqslant t \leqslant 3$.
(e) When is the particle speeding up? When is it slowing down?

74. The volume of a right circular cone is $V = \frac{1}{3}\pi r^2 h$, where r is the radius of the base and h is the height.
(a) Find the rate of change of the volume with respect to the height if the radius is constant.
(b) Find the rate of change of the volume with respect to the radius if the height is constant.

75. The mass of part of a wire is $x(1 + \sqrt{x})$ kilograms, where x is measured in meters from one end of the wire. Find the linear density of the wire when $x = 4$ m.

76. The cost, in dollars, of producing x units of a certain commodity is

$$C(x) = 920 + 2x - 0.02x^2 + 0.00007x^3$$

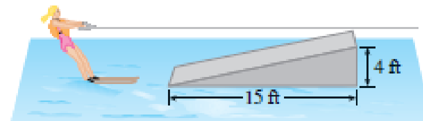
(a) Find the marginal cost function.
(b) Find $C'(100)$ and explain its meaning.
(c) Compare $C'(100)$ with the cost of producing the 101st item.

77. The volume of a cube is increasing at a rate of 10 cm^3/min. How fast is the surface area increasing when the length of an edge is 30 cm?

78. A paper cup has the shape of a cone with height 10 cm and radius 3 cm (at the top). If water is poured into the cup at a rate of 2 cm^3/s, how fast is the water level rising when the water is 5 cm deep?

79. A balloon is rising at a constant speed of 5 ft/s. A boy is cycling along a straight road at a speed of 15 ft/s. When he passes under the balloon, it is 45 ft above him. How fast is the distance between the boy and the balloon increasing 3 s later?

80. A waterskier skis over the ramp shown in the figure at a speed of 30 ft/s. How fast is she rising as she leaves the ramp?



81. The angle of elevation of the sun is decreasing at a rate of 0.25 rad/h. How fast is the shadow cast by a 400-ft-tall building increasing when the angle of elevation of the sun is $\pi/6$?

82. (a) Find the linear approximation to $f(x) = \sqrt{25 - x^2}$ near 3.
(b) Illustrate part (a) by graphing f and the linear approximation.
(c) For what values of x is the linear approximation accurate to within 0.1?

83. (a) Find the linearization of $f(x) = \sqrt[3]{1 + 3x}$ at $a = 0$. State the corresponding linear approximation and use it to give an approximate value for $\sqrt[3]{1.03}$.
(b) Determine the values of x for which the linear approximation given in part (a) is accurate to within 0.1.

84. Evaluate dy if $y = x^3 - 2x^2 + 1$, $x = 2$, and $dx = 0.2$.

85. A window has the shape of a square surmounted by a semicircle. The base of the window is measured as having width 60 cm with a possible error in measurement of 0.1 cm. Use differentials to estimate the maximum error possible in computing the area of the window.

86–88 Express the limit as a derivative and evaluate.

86. $\displaystyle\lim_{x \to 1} \frac{x^{17} - 1}{x - 1}$

87. $\displaystyle\lim_{h \to 0} \frac{\sqrt[4]{16 + h} - 2}{h}$

88. $\displaystyle\lim_{\theta \to \pi/3} \frac{\cos\theta - 0.5}{\theta - \pi/3}$

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89. Evaluate $\displaystyle\lim_{x \to 0} \frac{\sqrt{1 + \tan x} - \sqrt{1 + \sin x}}{x^3}$.

90. Suppose f is a differentiable function such that $f(g(x)) = x$ and $f'(x) = 1 + [f(x)]^2$. Show that $g'(x) = 1/(1 + x^2)$.

91. Find $f'(x)$ if it is known that

$$\frac{d}{dx}[f(2x)] = x^2$$

92. Show that the length of the portion of any tangent line to the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ cut off by the coordinate axes is constant.