Chapter 5

EXAMPLE 7 **Transforming an Equation from Polar to Rectangular Form**

Transform the equation $r = 6 \cos \theta$ from polar coordinates to rectangular coc nates, and identify the graph.

Solution If we multiply each side by r, it will be easier to apply formulas (1) and (2).

$$r = 6 \cos \theta$$
$$r^2 = 6r \cos \theta \quad \text{Multiply each side by } r.$$
$$x^2 + y^2 = 6x \quad r^2 = x^2 + y^2;\ x = r\cos\theta$$

This is the equation of a circle, so we proceed to complete the square to obtain standard form of the equation.

$$x^2 + y^2 = 6x$$
$$(x^2 - 6x) + y^2 = 0 \quad \text{General form}$$
$$(x^2 - 6x + 9) + y^2 = 9 \quad \text{Complete the square in } x.$$
$$(x - 3)^2 + y^2 = 9 \quad \text{Factor.}$$

This is the standard form of the equation of a circle with center $(3, 0)$ and radiı

Now Work PROBLEM 75

EXAMPLE 8 **Transforming an Equation from Rectangular to Polar Form**

Transform the equation $4xy = 9$ from rectangular coordinates to polar coordin;

Solution We use formula (1): $x = r \cos \theta$ and $y = r \sin \theta$.

$$4xy = 9$$
$$4(r\cos\theta)(r\sin\theta) = 9 \quad x = r\cos\theta,\ y = r\sin\theta$$
$$4r^2 \cos\theta \sin\theta = 9$$

This is the polar form of the equation. It can be simplified as shown next:

$$2r^2(2\sin\theta\cos\theta) = 9 \quad \text{Factor out } 2r^2.$$
$$2r^2 \sin(2\theta) = 9 \quad \text{Double-angle Formula}$$

Now Work PROBLEM 69

5.1 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red*

1. Plot the point whose rectangular coordinates are $(3, -1)$. (p. 2)
2. To complete the square of $x^2 + 6x$, add _____. (pp. A29–A30)
3. If $P = (x, y)$ is the point on the unit circle that correspon the angle θ, then $\sin \theta =$ _____. (pp. 107–109)
4. $\tan^{-1}(-1) =$ _____. (pp. 184–186)

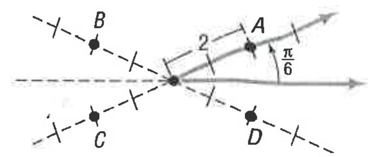
Concepts and Vocabulary

5. In polar coordinates, the origin is called the _____ and the positive x-axis is referred to as the __________.
6. Another representation in polar coordinates for the point $\left(2, \frac{\pi}{3}\right)$ is $\left(\_\_\_\_, \frac{4\pi}{3}\right)$.
7. The polar coordinates $\left(-2, \frac{\pi}{6}\right)$ are represented in rectangular coordinates by (_____, _____).
8. ***True or False*** The polar coordinates of a point are uni
9. ***True or False*** The rectangular coordinates of a poin unique.
10. ***True or False*** In (r, θ), the number r can be negative.

5.1

Skill Building

In Problems 11–18, match each point in polar coordinates with either A, B, C, or D on the graph.



11. $\left(2, -\frac{11\pi}{6}\right)$ **12.** $\left(-2, -\frac{\pi}{6}\right)$ **13.** $\left(-2, \frac{\pi}{6}\right)$ **14.** $\left(2, \frac{7\pi}{6}\right)$

15. $\left(2, \frac{5\pi}{6}\right)$ **16.** $\left(-2, \frac{5\pi}{6}\right)$ **17.** $\left(-2, \frac{7\pi}{6}\right)$ **18.** $\left(2, \frac{11\pi}{6}\right)$

In Problems 19–30, plot each point given in polar coordinates.

19. $(3, 90°)$ **20.** $(4, 270°)$ **21.** $(-2, 0)$ **22.** $(-3, \pi)$

23. $\left(6, \frac{\pi}{6}\right)$ **24.** $\left(5, \frac{5\pi}{3}\right)$ **25.** $(-2, 135°)$ **26.** $(-3, 120°)$

27. $\left(-1, -\frac{\pi}{3}\right)$ **28.** $\left(-3, -\frac{3\pi}{4}\right)$ **29.** $(-2, -\pi)$ **30.** $\left(-3, -\frac{\pi}{2}\right)$

In Problems 31–38, plot each point given in polar coordinates, and find other polar coordinates (r, θ) of the point for which:
(a) $r > 0,\ -2\pi \le \theta < 0$ *(b)* $r < 0,\ 0 \le \theta < 2\pi$ *(c)* $r > 0,\ 2\pi \le \theta < 4\pi$

31. $\left(5, \frac{2\pi}{3}\right)$ **32.** $\left(4, \frac{3\pi}{4}\right)$ **33.** $(-2, 3\pi)$ **34.** $(-3, 4\pi)$

35. $\left(1, \frac{\pi}{2}\right)$ **36.** $(2, \pi)$ **37.** $\left(-3, -\frac{\pi}{4}\right)$ **38.** $\left(-2, -\frac{2\pi}{3}\right)$

In Problems 39–54, the polar coordinates of a point are given. Find the rectangular coordinates of each point.

39. $\left(3, \frac{\pi}{2}\right)$ **40.** $\left(4, \frac{3\pi}{2}\right)$ **41.** $(-2, 0)$ **42.** $(-3, \pi)$

43. $(6, 150°)$ **44.** $(5, 300°)$ **45.** $\left(-2, \frac{3\pi}{4}\right)$ **46.** $\left(-2, \frac{2\pi}{3}\right)$

47. $\left(-1, -\frac{\pi}{3}\right)$ **48.** $\left(-3, -\frac{3\pi}{4}\right)$ **49.** $(-2, -180°)$ **50.** $(-3, -90°)$

51. $(7.5, 110°)$ **52.** $(-3.1, 182°)$ **53.** $(6.3, 3.8)$ **54.** $(8.1, 5.2)$

In Problems 55–66, the rectangular coordinates of a point are given. Find polar coordinates for each point.

55. $(3, 0)$ **56.** $(0, 2)$ **57.** $(-1, 0)$ **58.** $(0, -2)$

59. $(1, -1)$ **60.** $(-3, 3)$ **61.** $\left(\sqrt{3}, 1\right)$ **62.** $\left(-2, -2\sqrt{3}\right)$

63. $(1.3, -2.1)$ **64.** $(-0.8, -2.1)$ **65.** $(8.3, 4.2)$ **66.** $(-2.3, 0.2)$

In Problems 67–74, the letters x and y represent rectangular coordinates. Write each equation using polar coordinates (r, θ).

67. $2x^2 + 2y^2 = 3$ **68.** $x^2 + y^2 = x$ **69.** $x^2 = 4y$ **70.** $y^2 = 2x$

71. $2xy = 1$ **72.** $4x^2y = 1$ **73.** $x = 4$ **74.** $y = -3$

In Problems 75–82, the letters r and θ represent polar coordinates. Write each equation using rectangular coordinates (x, y).

75. $r = \cos\theta$ **76.** $r = \sin\theta + 1$ **77.** $r^2 = \cos\theta$ **78.** $r = \sin\theta - \cos\theta$

79. $r = 2$ **80.** $r = 4$ **81.** $r = \frac{4}{1 - \cos\theta}$ **82.** $r = \frac{3}{3 - \cos\theta}$

Applications and Extensions

83. Chicago In Chicago, the road system is set up like a Cartesian plane, where streets are indicated by the number of blocks they are from Madison Street and State Street. For example, Wrigley Field in Chicago is located at 1060 West Addison, which is 10 blocks west of State Street and 36 blocks north of Madison Street. Treat the intersection of Madison Street and State Street as the origin of a coordinate system, with East being the positive x-axis.

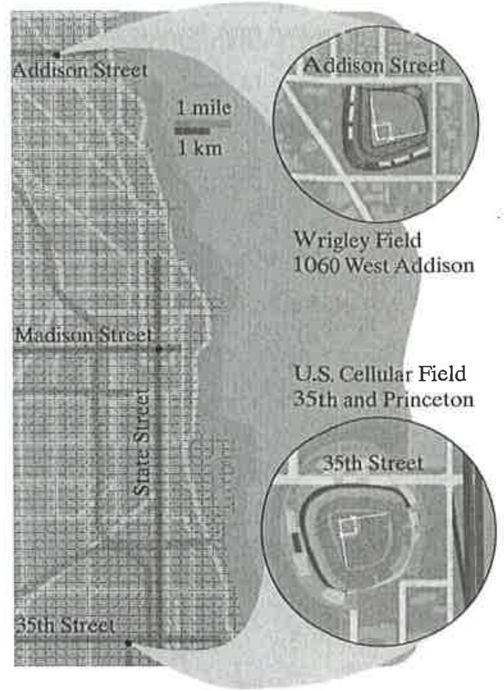
(a) Write the location of Wrigley Field using rectangular coordinates.

(b) Write the location of Wrigley Field using polar coordinates. Use the East direction for the polar axis. Express θ in degrees.

(c) U.S. Cellular Field, home of the White Sox, is located at 35th and Princeton, which is 3 blocks west of State Street and 35 blocks south of Madison. Write the location of U.S. Cellular Field using rectangular coordinates.

(d) Write the location of U.S. Cellular Field using polar coordinates. Use the East direction for the polar axis. Express θ in degrees.

City of Chicago, Illinois



84. Show that the formula for the distance d between two points $P_1 = (r_1, \theta_1)$ and $P_2 = (r_2, \theta_2)$ is

$$d = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2 - \theta_1)}$$

Discussion and Writing

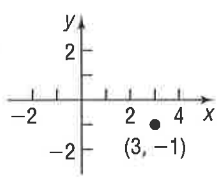
85. In converting from polar coordinates to rectangular coordinates, what formulas will you use?

86. Explain how you proceed to convert from rectangular coordinates to polar coordinates.

87. Is the street system in your town based on a rectangular ordinate system, a polar coordinate system, or some ot system? Explain.

'Are You Prepared?' Answers

1.



2. 9 **3.** y **4.** $-\frac{\pi}{4}$

5.2 Polar Equations and Graphs

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Symmetry (Section 1.2, pp. 12–13)
- Circles (Section 1.2, pp. 15–18)
- Even–Odd Properties of Trigonometric Functions (Section 2.3, pp. 132–133)
- Difference Formulas for Sine and Cosine (Section 3.4, pp. 203 and 206)
- Value of the Sine and Cosine Functions at Certain Angles (Section 2.2, pp. 112 and 115)

Now Work the 'Are You Prepared?' problems on page 321.

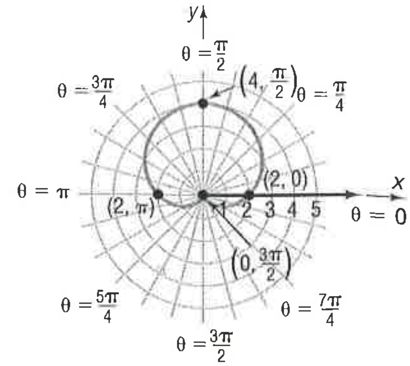
OBJECTIVES
1 Graph and Identify Polar Equations by Converting to Rectangular Equations (p. 309)
2 Test Polar Equations for Symmetry (p. 313)
3 Graph Polar Equations by Plotting Points (p. 314)

Just as a rectangular grid may be used to plot points given by rectangular coordina as in Figure 18(a), we can use a grid consisting of concentric circles (with center the pole) and rays (with vertices at the pole) to plot points given by polar coordina as shown in Figure 18(b). We shall use such **polar grids** to graph *polar equations*.

Table 8

θ	$r = 2 + 2\sin\theta$
0	$2 + 2(0) = 2$
$\frac{\pi}{2}$	$2 + 2(1) = 4$
π	$2 + 2(0) = 2$
$\frac{3\pi}{2}$	$2 + 2(-1) = 0$
2π	$2 + 2(0) = 2$

Figure 32



Calculus Comment For those of you who are planning to study calculus, a comment about one important role of polar equations is in order.

In rectangular coordinates, the equation $x^2 + y^2 = 1$, whose graph is the unit circle, is not the graph of a function. In fact, it requires two functions to obtain the graph of the unit circle:

$$y_1 = \sqrt{1 - x^2} \quad \text{Upper semicircle} \qquad y_2 = -\sqrt{1 - x^2} \quad \text{Lower semicircle}$$

In polar coordinates, the equation $r = 1$, whose graph is also the unit circle, does define a function. That is, for each choice of θ there is only one corresponding value of r, that is, $r = 1$. Since many problems in calculus require the use of functions, the opportunity to express nonfunctions in rectangular coordinates as functions in polar coordinates becomes extremely useful.

Note also that the vertical-line test for functions is valid only for equations in rectangular coordinates.

Historical Feature

Jakob Bernoulli (1654–1705)

Polar coordinates seem to have been invented by Jakob Bernoulli (1654–1705) in about 1691, although, as with most such ideas, earlier traces of the notion exist. Early users of calculus remained committed to rectangular coordinates, and polar coordinates did not become widely used until the early 1800s. Even then, it was mostly geometers who used them for describing odd curves. Finally, about the mid-1800s, applied mathematicians realized the tremendous simplification that polar coordinates make possible in the description of objects with circular or cylindrical symmetry. From then on their use became widespread.

5.2 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. If the rectangular coordinates of a point are $(4, -6)$, the point symmetric to it with respect to the origin is _____. (pp. 12–13)
2. The difference formula for cosine is $\cos(A - B) =$ _____. (p. 203)
3. The standard equation of a circle with center at $(-2, 5)$ and radius 3 is _____. (pp. 15–18)
4. Is the sine function even, odd, or neither? (pp. 132–133)
5. $\sin\frac{5\pi}{4} =$ _____. (p. 115)
6. $\cos\frac{2\pi}{3} =$ _____. (p. 115)

5.2

Concepts and Vocabulary

7. An equation whose variables are polar coordinates is called a(n) _____ _____.
8. Using polar coordinates (r, θ), the circle $x^2 + y^2 = 2x$ takes the form _____.
9. A polar equation is symmetric with respect to the pole if an equivalent equation results when r is replaced by _____.
10. ***True or False*** The tests for symmetry in polar coordin are necessary, but not sufficient.
11. ***True or False*** The graph of a cardioid never passes thro the pole.
12. ***True or False*** All polar equations have a symmetric fea

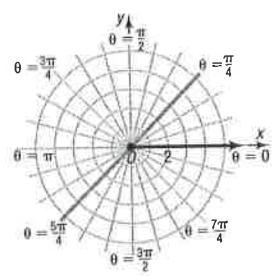
Skill Building

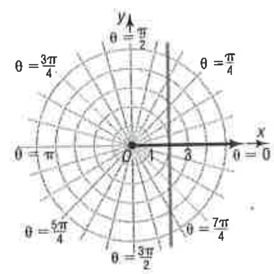
In Problems 13–28, transform each polar equation to an equation in rectangular coordinates. Then identify and graph the equatio

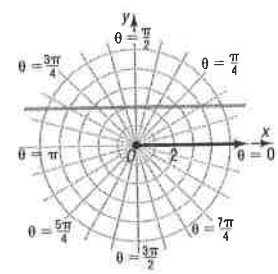
13. $r = 4$
14. $r = 2$
15. $\theta = \frac{\pi}{3}$
16. $\theta = -\frac{\pi}{4}$
17. $r \sin\theta = 4$
18. $r \cos\theta = 4$
19. $r \cos\theta = -2$
20. $r \sin\theta = -2$
21. $r = 2\cos\theta$
22. $r = 2\sin\theta$
23. $r = -4\sin\theta$
24. $r = -4\cos\theta$
25. $r \sec\theta = 4$
26. $r \csc\theta = 8$
27. $r \csc\theta = -2$
28. $r \sec\theta = -4$

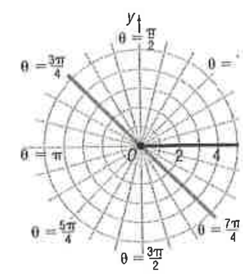
In Problems 29–36, match each of the graphs (A) through (H) to one of the following polar equations.

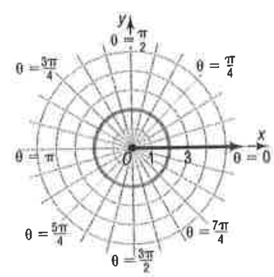
29. $r = 2$
30. $\theta = \frac{\pi}{4}$
31. $r = 2\cos\theta$
32. $r \cos\theta = 2$
33. $r = 1 + \cos\theta$
34. $r = 2\sin\theta$
35. $\theta = \frac{3\pi}{4}$
36. $r \sin\theta = 2$

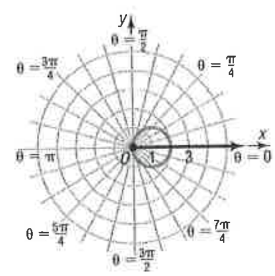

(A)

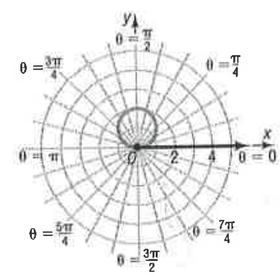

(B)

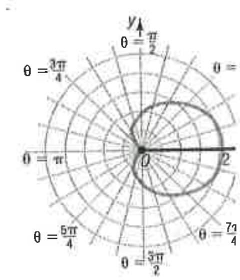

(C)


(D)


(E)


(F)


(G)


(H)

In Problems 37–60, identify and graph each polar equation.

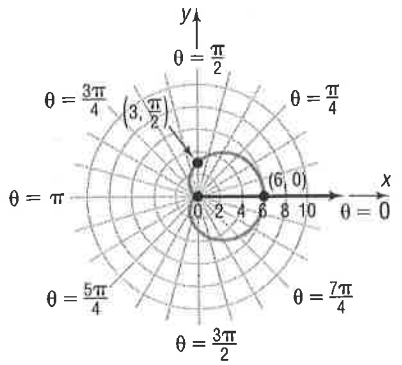
37. $r = 2 + 2\cos\theta$
38. $r = 1 + \sin\theta$
39. $r = 3 - 3\sin\theta$
40. $r = 2 - 2\cos\theta$
41. $r = 2 + \sin\theta$
42. $r = 2 - \cos\theta$
43. $r = 4 - 2\cos\theta$
44. $r = 4 + 2\sin\theta$
45. $r = 1 + 2\sin\theta$
46. $r = 1 - 2\sin\theta$
47. $r = 2 - 3\cos\theta$
48. $r = 2 + 4\cos\theta$
49. $r = 3\cos(2\theta)$
50. $r = 2\sin(3\theta)$
51. $r = 4\sin(5\theta)$
52. $r = 3\cos(4\theta)$
53. $r^2 = 9\cos(2\theta)$
54. $r^2 = \sin(2\theta)$
55. $r = 2^\theta$
56. $r = 3^\theta$
57. $r = 1 - \cos\theta$
58. $r = 3 + \cos\theta$
59. $r = 1 - 3\cos\theta$
60. $r = 4\cos(3\theta)$

5.2

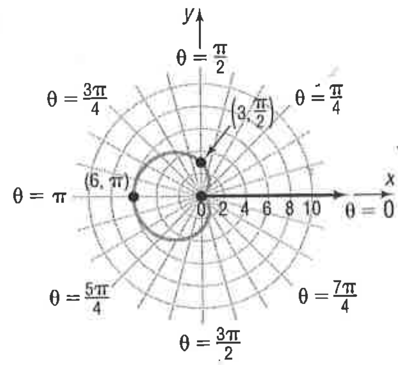
Applications and Extensions

In Problems 61–64, the polar equation for each graph is either $r = a + b\cos\theta$ or $r = a + b\sin\theta$, $a > 0$, $b > 0$. Select the correct equation and find the values of a and b.

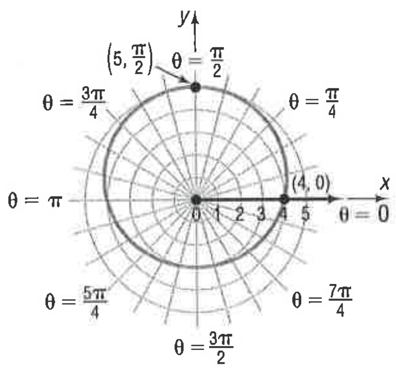
61.



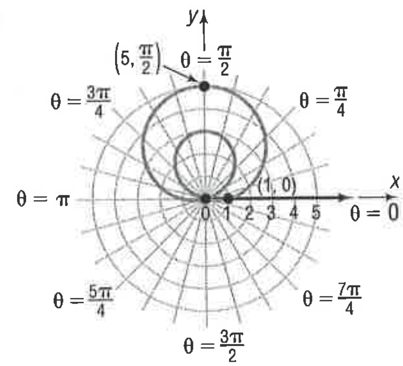
62.



63.



64.



In Problems 65–74, graph each polar equation.

65. $r = \dfrac{2}{1 - \cos\theta}$ *(parabola)*

66. $r = \dfrac{2}{1 - 2\cos\theta}$ *(hyperbola)*

67. $r = \dfrac{1}{3 - 2\cos\theta}$ *(ellipse)*

68. $r = \dfrac{1}{1 - \cos\theta}$ *(parabola)*

69. $r = \theta, \quad \theta \geq 0$ *(spiral of Archimedes)*

70. $r = \dfrac{3}{\theta}$ *(reciprocal spiral)*

71. $r = \csc\theta - 2, \quad 0 < \theta < \pi$ *(conchoid)*

72. $r = \sin\theta\tan\theta$ *(cissoid)*

73. $r = \tan\theta, \quad -\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$ *(kappa curve)*

74. $r = \cos\dfrac{\theta}{2}$

75. Show that the graph of the equation $r\sin\theta = a$ is a horizontal line a units above the pole if $a > 0$ and $|a|$ units below the pole if $a < 0$.

76. Show that the graph of the equation $r\cos\theta = a$ is a vertical line a units to the right of the pole if $a > 0$ and $|a|$ units to the left of the pole if $a < 0$.

77. Show that the graph of the equation $r = 2a\sin\theta$, $a > 0$, is a circle of radius a with center at $(0, a)$ in rectangular coordinates.

78. Show that the graph of the equation $r = -2a\sin\theta$, $a > 0$, is a circle of radius a with center at $(0, -a)$ in rectangular coordinates.

79. Show that the graph of the equation $r = 2a\cos\theta$, $a > 0$, is a circle of radius a with center at $(a, 0)$ in rectangular coordinates.

80. Show that the graph of the equation $r = -2a\cos\theta$, $a > 0$, is a circle of radius a with center at $(-a, 0)$ in rectangular coordinates.

not coincidental. In fact, you are asked to show that these results hold for comj nth roots in Problems 63 through 65.

Now Work PROBLEM 53

Historical Feature

John Wallis

The Babylonians, Greeks, and Arabs considered square roots of negative quantities to be impossible and equations with complex solutions to be unsolvable. The first hint that there was some connection between real solutions of equations and complex numbers came when Girolamo Cardano (1501–1576) and Tartaglia (1499–1557) found *real* roots of cubic equations by taking cube roots of *complex* quantities. For centuries thereafter, mathematicians worked with complex numbers without much belief in their actu: istence. In 1673, John Wallis appears to have been the first to su(the graphical representation of complex numbers, a truly significan that was not pursued further until about 1800. Several people, in ing Karl Friedrich Gauss (1777–1855), then rediscovered the idea graphical representation helped to establish complex numbers as members of the number family. In practical applications, complex bers have found their greatest uses in the study of alternating cu where they are a commonplace tool, and in the field of suba physics.

Historical Problems

1. The quadratic formula will work perfectly well if the coefficients are complex numbers. Solve the following using De Moivre's Theorem \ necessary.
   [Hint: The answers are "nice."]
   (a) $z^2 - (2 + 5i)z - 3 + 5i = 0$ (b) $z^2 - (1 + i)z - 2 - i = 0$

5.3 Assess your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in re*

1. The conjugate of $-4 - 3i$ is _____. (pp. A36–A40)
2. The sum formula for the sine function is $\sin(A + B) =$ _____. (p. 206)
3. The sum formula for the cosine function is $\cos(A +$ _____. (p. 203)
4. $\sin 120° =$ _____; $\cos 240° =$ _____. (pp. 111–117)

Concepts and Vocabulary

5. When a complex number z is written in the polar form $z = r(\cos\theta + i\sin\theta)$, the nonnegative number r is the _____ or _____ of z, and the angle θ, $0 \le \theta < 2\pi$, is the _____ of z.
6. _____ Theorem can be used to raise a complex number to a power.
7. Every non-zero complex number will have exactly _____ cube roots.
8. ***True or False*** De Moivre's Theorem is useful for rai complex number to a positive integer power.
9. ***True or False*** Using De Moivre's Theorem, the squa complex number will have two answers.
10. ***True or False*** The polar form of a complex num unique.

Skill Building

In Problems 11–22, plot each complex number in the complex plane and write it in polar form. Express the argument in degrees

11. $1 + i$ **12.** $-1 + i$ **13.** $\sqrt{3} - i$ **14.** $1 - \sqrt{3}i$

15. $-3i$ **16.** -2 **17.** $4 - 4i$ **18.** $9\sqrt{3} + 9i$

19. $3 - 4i$ **20.** $2 + \sqrt{3}i$ **21.** $-2 + 3i$ **22.** $\sqrt{5} - i$

In Problems 23–32, write each complex number in rectangular form.

23. $2(\cos 120° + i\sin 120°)$ **24.** $3(\cos 210° + i\sin 210°)$ **25.** $4\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$

26. $2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

27. $3\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$

28. $4\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

29. $0.2(\cos 100° + i\sin 100°)$

30. $0.4(\cos 200° + i\sin 200°)$

31. $2\left(\cos\frac{\pi}{18} + i\sin\frac{\pi}{18}\right)$

32. $3\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)$

In Problems 33–40, find zw and $\frac{z}{w}$. Leave your answers in polar form.

33. $z = 2(\cos 40° + i\sin 40°)$
$w = 4(\cos 20° + i\sin 20°)$

34. $z = \cos 120° + i\sin 120°$
$w = \cos 100° + i\sin 100°$

35. $z = 3(\cos 130° + i\sin 130°)$
$w = 4(\cos 270° + i\sin 270°)$

36. $z = 2(\cos 80° + i\sin 80°)$
$w = 6(\cos 200° + i\sin 200°)$

37. $z = 2\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)$
$w = 2\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)$

38. $z = 4\left(\cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8}\right)$
$w = 2\left(\cos\frac{9\pi}{16} + i\sin\frac{9\pi}{16}\right)$

39. $z = 2 + 2i$
$w = \sqrt{3} - i$

40. $z = 1 - i$
$w = 1 - \sqrt{3}i$

In Problems 41–52, write each expression in the standard form $a + bi$.

41. $[4(\cos 40° + i\sin 40°)]^3$

42. $[3(\cos 80° + i\sin 80°)]^3$

43. $\left[2\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)\right]^5$

44. $\left[\sqrt{2}\left(\cos\frac{5\pi}{16} + i\sin\frac{5\pi}{16}\right)\right]^4$

45. $[\sqrt{3}(\cos 10° + i\sin 10°)]^6$

46. $\left[\frac{1}{2}(\cos 72° + i\sin 72°)\right]^5$

47. $\left[\sqrt{5}\left(\cos\frac{3\pi}{16} + i\sin\frac{3\pi}{16}\right)\right]^4$

48. $\left[\sqrt{3}\left(\cos\frac{5\pi}{18} + i\sin\frac{5\pi}{18}\right)\right]^6$

49. $(1 - i)^5$

50. $(\sqrt{3} - i)^6$

51. $(\sqrt{2} - i)^6$

52. $(1 - \sqrt{5}i)^8$

In Problems 53–60, find all the complex roots. Leave your answers in polar form with the argument in degrees.

53. The complex cube roots of $1 + i$

54. The complex fourth roots of $\sqrt{3} - i$

55. The complex fourth roots of $4 - 4\sqrt{3}i$

56. The complex cube roots of $-8 - 8i$

57. The complex fourth roots of $-16i$

58. The complex cube roots of -8

59. The complex fifth roots of i

60. The complex fifth roots of $-i$

Applications and Extensions

61. Find the four complex fourth roots of unity (1) and plot them.

62. Find the six complex sixth roots of unity (1) and plot them.

63. Show that each complex nth root of a nonzero complex number w has the same magnitude.

64. Use the result of Problem 63 to draw the conclusion that each complex nth root lies on a circle with center at the origin. What is the radius of this circle?

65. Refer to Problem 64. Show that the complex nth roots of a nonzero complex number w are equally spaced on the circle.

66. Prove formula (6).

'Are You Prepared?' Answers

1. $-4 + 3i$ **2.** $\sin A\cos B + \cos A\sin B$ **3.** $\cos A\cos B - \sin A\sin B$ **4.** $\frac{\sqrt{3}}{2}; -\frac{1}{2}$