Chapter 4

4.1

4.1 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in rea*

1. In a right triangle, if the length of the hypotenuse is 5 and the length of one of the other sides is 3, what is the length of the third side? (pp. A14–A15)
2. If θ is an acute angle, solve the equation $\tan\theta = \frac{1}{2}$. Express your answer in degrees, rounded to one decimal place. (pp. 225–229)
3. If θ is an acute angle, solve the equation $\sin\theta$ (pp. 225–229)

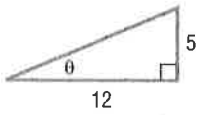
Concepts and Vocabulary

4. ***True or False*** $\sin 52° = \cos 48°$.
5. ***True or False*** In a right triangle, one of the angles is 90° and the sum of the other two angles is 90°.
6. When you look up at an object, the acute angle measured from the horizontal to a line-of-sight observation of the object is called the _____ _____ _____.
7. ***True or False*** In a right triangle, if two sides are known can solve the triangle.
8. ***True or False*** In a right triangle, if we know the two a angles, we can solve the triangle.

Skill Building

In Problems 9–18, find the exact value of the six trigonometric functions of the angle θ in each figure.

9.



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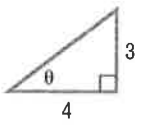

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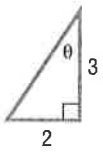

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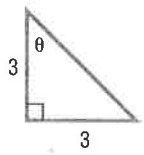

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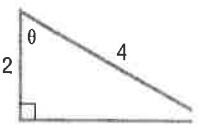

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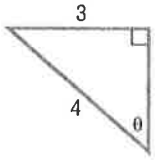

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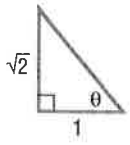

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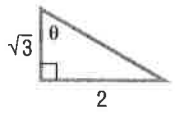

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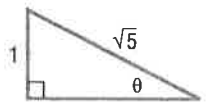

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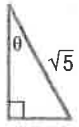


In Problems 19–28, find the exact value of each expression. Do not use a calculator.

19. $\sin 38° - \cos 52°$

20. $\tan 12° - \cot 78°$

21. $\dfrac{\cos 10°}{\sin 80°}$

22. $\dfrac{\cos 40°}{\sin 50°}$

23. $1 - \cos^2 20° - \cos^2 70°$

24. $1 + \tan^2 5° - \csc^2 85°$

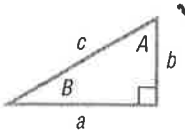
25. $\tan 20° - \dfrac{\cos 70°}{\cos 20°}$

26. $\cot 40° - \dfrac{\sin 5(}{\sin 4(}$

27. $\cos 35° \sin 55° + \sin 35° \cos 55°$

28. $\sec 35° \csc 55° - \tan 35° \cot 55°$

In Problems 29–42, use the right triangle shown below. Then, using the given information, solve the triangle.



29. $b = 5$, $B = 20°$; find a, c, and A

30. $b = 4$, $B = 10°$; find a, c, and A

31. $a = 6$, $B = 40°$; find b, c, and A

32. $a = 7$, $B = 50°$; find b, c, and A

33. $b = 4$, $A = 10°$; find a, c, and B

34. $b = 6$, $A = 20°$; find a, c, and B

35. $a = 5$, $A = 25°$; find b, c, and B

36. $a = 6$, $A = 40°$; find b, c, and B

37. $c = 9$, $B = 20°$; find b, a, and A

38. $c = 10$, $A = 40°$; find b, a, and B

39. $a = 5$, $b = 3$; find c, A, and B

40. $a = 2$, $b = 8$; find c, A, and B

41. $a = 2$, $c = 5$; find b, A, and B

42. $b = 4$, $c = 6$; find a, A, and B

Applications and Extensions

43. Geometry The hypotenuse of a right triangle is 5 inches. If one leg is 2 inches, find the degree measure of each angle.

44. Geometry The hypotenuse of a right triangle is 3 feet. If one leg is 1 foot, find the degree measure of each angle.

45. Geometry A right triangle has a hypotenuse of length 8 inches. If one angle is 35°, find the length of each leg.

46. Geometry A right triangle has a hypotenuse of length 10 centimeters. If one angle is 40°, find the length of each leg.

47. Geometry A right triangle contains a 25° angle.

(a) If one leg is of length 5 inches, what is the length of the hypotenuse?

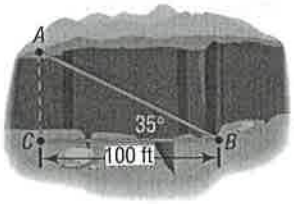
(b) There are two answers. How is this possible?

48. Geometry A right triangle contains an angle of $\frac{\pi}{8}$ radian.

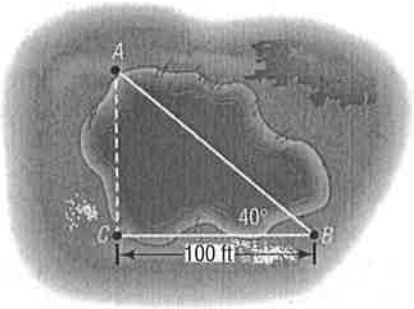
(a) If one leg is of length 3 meters, what is the length of the hypotenuse?

(b) There are two answers. How is this possible?

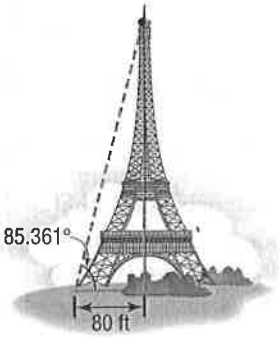
49. Finding the Width of a Gorge Find the distance from A to C across the gorge illustrated in the figure.



50. Finding the Distance across a Pond Find the distance from A to C across the pond illustrated in the figure.



51. The Eiffel Tower The tallest tower built before the era of television masts, the Eiffel Tower was completed on March 31, 1889. Find the height of the Eiffel Tower (before a television mast was added to the top) using the information given in the illustration.



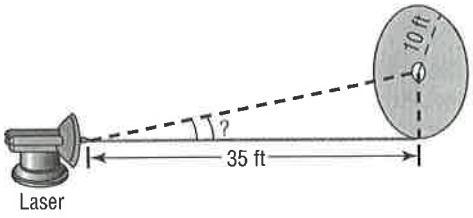
52. Finding the Distance of a Ship from Shore A person in a small boat, offshore from a vertical cliff known to be 100 feet in height, takes a sighting of the top of the cliff. If the angle of elevation is found to be 25°, how far offshore is the boat?

53. Finding the Distance to a Plateau Suppose that you are headed toward a plateau 50 meters high. If the angle of elevation to the top of the plateau is 20°, how far are you from the base of the plateau?

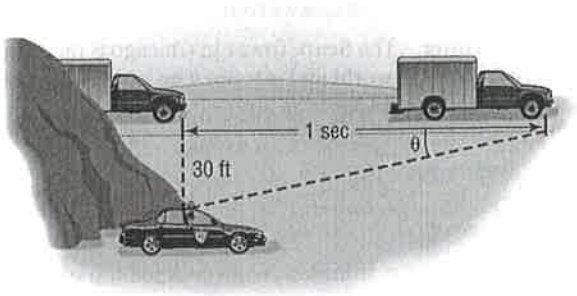
54. Finding the Reach of a Ladder A 22-foot extension ladder leaning against a building makes a 70° angle with the ground. How far up the building does the ladder touch?

55. Finding the Angle of Elevation of the Sun At 10 AM on April 26, 2006, a building 300 feet high casts a shadow 50 feet long. What is the angle of elevation of the Sun?

56. Directing a Laser Beam A laser beam is to be directed through a small hole in the center of a circle of radius 10 feet. The origin of the beam is 35 feet from the circle (see the figure). At what angle of elevation should the beam be aimed to ensure that it goes through the hole?



57. Finding the Speed of a Truck A state trooper is hidden 30 feet from a highway. One second after a truck passes, the angle θ between the highway and the line of observation from the patrol car to the truck is measured. See the illustration.



(a) If the angle measures 15°, how fast is the truck traveling? Express the answer in feet per second and in miles per hour.

(b) If the angle measures 20°, how fast is the truck traveling? Express the answer in feet per second and in miles per hour.

(c) If the speed limit is 55 miles per hour and a speeding ticket is issued for speeds of 5 miles per hour or more over the limit, for what angles should the trooper issue a ticket?

58. Security A security camera in a neighborhood bank is mounted on a wall 9 feet above the floor. What angle of

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depression should be used if the camera is to be directed to a spot 6 feet above the floor and 12 feet from the wall?

59. Finding the Length of a Guy Wire A radio transmission tower is 200 feet high. How long should a guy wire be if it is to be attached to the tower 10 feet from the top and is to make an angle of 69° with the ground?

60. Finding the Height of a Tower A guy wire 80 feet long is attached to the top of a radio transmission tower, making an angle of 65° with the ground. How high is the tower?

61. Washington Monument The angle of elevation of the Sun is 35.1° at the instant the shadow cast by the Washington Monument is 789 feet long. Use this information to calculate the height of the monument.

62. Finding the Length of a Mountain Trail A straight trail with an inclination of 17° leads from a hotel at an elevation of 9000 feet to a mountain lake at an elevation of 11,200 feet. What is the length of the trail?

63. Finding the Bearing of an Aircraft A DC-9 aircraft leaves Midway Airport from runway 4 RIGHT, whose bearing is N40°E. After flying for $\frac{1}{2}$ mile, the pilot requests permission to turn 90° and head toward the southeast. The permission is granted. After the airplane goes 1 mile in this direction, what bearing should the control tower use to locate the aircraft?

64. Finding the Bearing of a Ship A ship leaves the port of Miami with a bearing of S80°E and a speed of 15 knots. After 1 hour, the ship turns 90° toward the south. After 2 hours, maintaining the same speed, what is the bearing to the ship from port?

65. Niagara Falls Incline Railway Situated between Portage Road and the Niagara Parkway directly across from the Canadian Horseshoe Falls, the Falls Incline Railway is a funicular that carries passengers up an embankment to Table Rock Observation Point. If the length of the track is 51.8 meters and the angle of inclination is 36°2′, determine the height of the embankment.

Source: www.niagaraparks.com

66. Sears Tower The Sears Tower in Chicago is the third tallest building in the world and is topped by a high antenna. A surveyor on the ground makes the following measurement:

1. The angle of elevation from his position to the top of the building is 34°.
2. The distance from his position to the top of the building is 2593 feet.
3. The distance from his position to the top of the antenna is 2743 feet.

(a) How far away from the (base of the) building is the surveyor located?

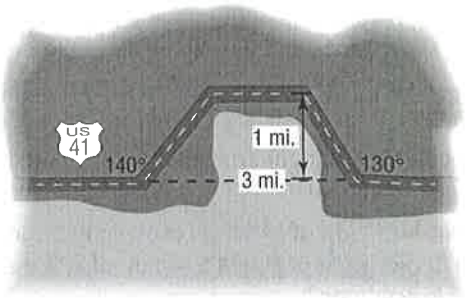
(b) How tall is the building?

(c) What is the angle of elevation from the surveyor to the top of the antenna?

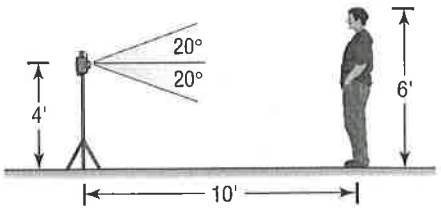
(d) How tall is the antenna?

Source: www.infoplease.com/ce6/us/A0844218.html

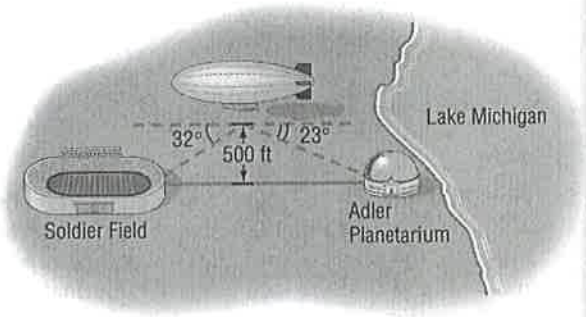
67. Constructing a Highway A highway whose primary directions are north–south is being constructed along the west coast of Florida. Near Naples, a bay obstructs the straight path of the road. Since the cost of a bridge is prohibitive, engineers decide to go around the bay. The illustration shows the path that they decide on and the measurements taken. What is the length of highway needed to go around the bay?



68. Photography A camera is mounted on a tripod 4 feet high at a distance of 10 feet from George, who is 6 feet tall. See the illustration. If the camera lens has angles of depression and elevation of 20°, will George's feet and head be seen by the lens? If not, how far back will the camera need to be moved to include George's feet and head?



69. Finding the Distance between Two Objects A blimp, suspended in the air at a height of 500 feet, lies directly over a line from Soldier Field to the Adler Planetarium on Lake Michigan (see the figure). If the angle of depression from the blimp to the stadium is 32° and from the blimp to the planetarium is 23°, find the distance between Soldier Field and the Adler Planetarium.



70. Hot-Air Balloon While taking a ride in a hot-air balloon in Napa Valley, Francisco wonders how high he is. To find out, he chooses a landmark that is to the east of the balloon and measures the angle of depression to be 54°. A few minutes later, after traveling 100 feet east, the angle of depression to the same landmark is determined to be 61°. Use this information to determine the height of the balloon.

71. Mt. Rushmore To measure the height of Lincoln's caricature on Mt. Rushmore, two sightings 800 feet from the base of the mountain are taken. If the angle of elevation to the

bottom of Lincoln's face is 32° and the angle of elevation to the top is 35°, what is the height of Lincoln's face?

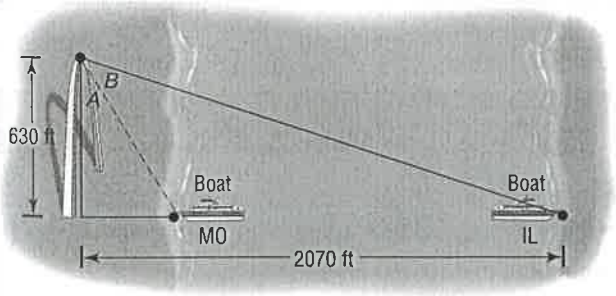
72. **The CN Tower** The CN Tower, located in Toronto, Canada, is the tallest structure in the world. While visiting Toronto, a tourist wondered what the height of the tower above the top of the Sky Pod is. While standing 4000 feet from the tower, she measured the angle to the top of the Sky Pod to be 20.1°. At this same distance, the angle of elevation to the top of the tower was found to be 24.4°. Use this information to determine the height of the tower above the Sky Pod.

73. **Chicago Skyscrapers** The angle of inclination from the base of the John Hancock Center to the top of the main structure of the Sears Tower is approximately 10.3°. If the main structure of the Sears Tower is 1451 feet tall, how far apart are the two skyscrapers? Assume the bases of the two buildings are at the same elevation.

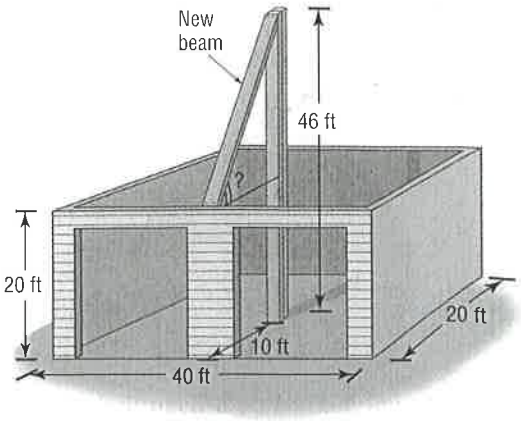
Source: www.emporis.com

74. **Estimating the Width of the Mississippi River** A tourist at the top of the Gateway Arch (height, 630 feet) in St. Louis, Missouri observes a boat moored on the Illinois side of the Mississippi River 2070 feet directly across from the Arch. She also observes a boat moored on the Missouri side directly across from the first boat (see diagram). Given that $B = \cot^{-1}\frac{67}{55}$, estimate the width of the Mississippi River at the St. Louis riverfront.

Source: U.S. Army Corps of Engineers

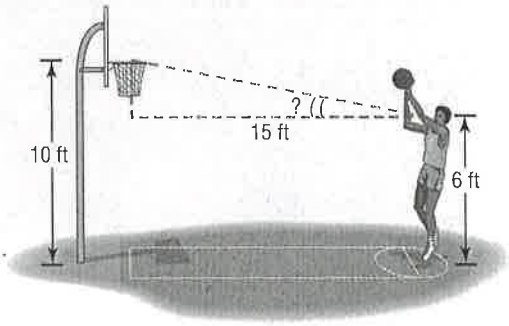


75. **Finding the Pitch of a Roof** A carpenter is preparing to put a roof on a garage that is 20 feet by 40 feet by 20 feet. A steel support beam 46 feet in length is positioned in the center of the garage. To support the roof, another beam will be attached to the top of the center beam (see the figure). At what angle of elevation is the new beam? In other words, what is the pitch of the roof?

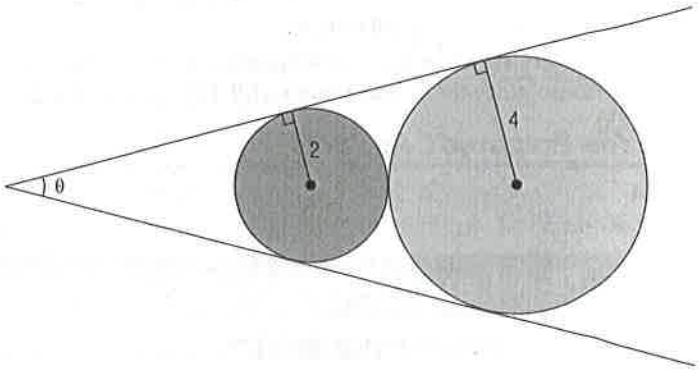


76. **Shooting Free Throws in Basketball** The eyes of a basketball player are 6 feet above the floor. The player is at the free-throw line, which is 15 feet from the center of the basket rim (see the figure). What is the angle of elevation from the player's eyes to the center of the rim?

[**Hint:** The rim is 10 feet above the floor.]



77. **Geometry** Find the value of the angle θ in degrees rounded to the nearest tenth of a degree.



78. **Surveillance Satellites** A surveillance satellite circles Earth at a height of h miles above the surface. Suppose that d is the distance, in miles, on the surface of Earth that can be observed from the satellite. See the illustration on page 508.

(a) Find an equation that relates the central angle θ to the height h.

(b) Find an equation that relates the observable distance d and θ.

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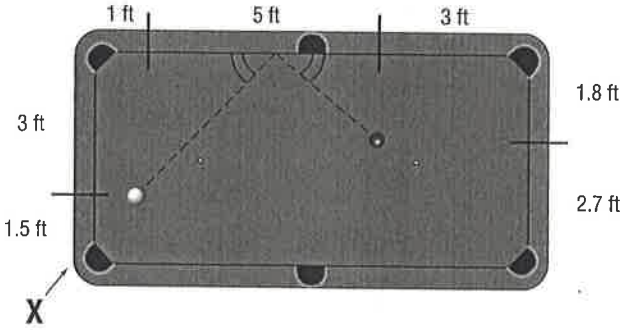
(c) Find an equation that relates d and h.

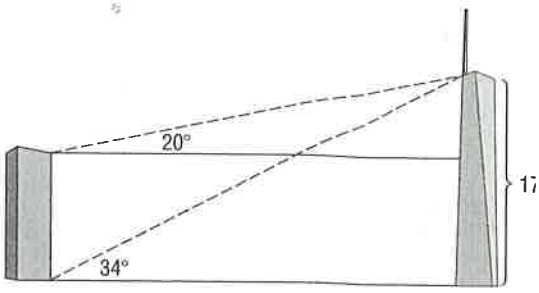
(d) If d is to be 2500 miles, how high must the satellite orbit above Earth?

(e) If the satellite orbits at a height of 300 miles, what distance d on the surface can be observed?

79. Calculating Pool Shots A pool player located at **X** wants to shoot the white ball off the top cushion and hit the red ball dead center. He knows from physics that the white ball will come off a cushion at the same angle as it hits a cushion. Where on the top cushion should he hit the white ball?



80. The Freedom Tower The Freedom Tower is to be terpiece of the rebuilding of the World Trade Center York City. The tower will be 1776 feet tall (not inc broadcast antenna). The angle of elevation from the an office building to the top of the tower is 34°. The elevation from the helipad on the roof of the office b to the top of the tower is 20°.



(a) How far away is the office building from the F Tower? Assume the side of the tower is vertical. to the nearest foot.

(b) How tall is the office building? Round to the near

Discussion and Writing

81. Explain how you would measure the width of the Grand Canyon from a point on its ridge.

82. Explain how you would measure the height of a TV tower that is on the roof of a tall building.

83. The Gibb's Hill Lighthouse. Southampton, Bermuda In operation since 1846, the Gibb's Hill Lighthouse stands 117 feet high on a hill 245 feet high, so its beam of 362 feet above sea level. A brochure states that ships 4 away can see the light and planes flying at 10,000 feet it 120 miles away. Verify the accuracy of these state What assumption did the brochure make about the he the ship?

'Are You Prepared?' Answers

1. 4 **2.** 26.6° **3.** 30°

4.2 The Law of Sines

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Trigonometric Equations (I) (Section 3.7, pp. 225–229)
- Difference Formula for the Sine Function (Section 3.4, p. 206)
- Geometry Essentials (Appendix A, Section A.2, pp. A14–A19)

Now Work the 'Are You Prepared?' problems on page 265.

OBJECTIVES 1 Solve SAA or ASA Triangles (p. 260)
2 Solve SSA Triangles (p. 261)
3 Solve Applied Problems (p. 263)

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which again gives

$$h = c \sin A$$

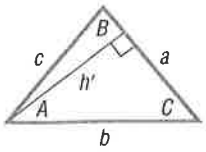
So, whether the triangle has three acute angles or has two acute angles and one obtuse angle, equations (3) and (4) hold. As a result, we may equate the expressions for h in equations (3) and (4) to get

$$a \sin C = c \sin A$$

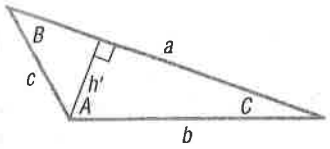
from which

$$\frac{\sin A}{a} = \frac{\sin C}{c} \tag{5}$$

Figure 34



(a)



(b)

In a similar manner, by constructing the altitude h' from the vertex of angle A as shown in Figures 34(a) and (b), we can show that

$$\sin B = \frac{h'}{c} \quad \text{and} \quad \sin C = \frac{h'}{b}$$

Equating the expressions for h', we find that

$$h' = c \sin B = b \sin C$$

from which

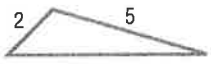
$$\frac{\sin B}{b} = \frac{\sin C}{c} \tag{6}$$

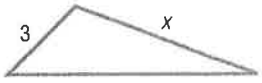
When equations (5) and (6) are combined, we have equation (1), the Law of Sines. ■

4.2 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. The difference formula for the sine function is $\sin(A - B) =$ _____. (p. 206)
2. If θ is an acute angle, solve the equation $\cos\theta = \frac{\sqrt{3}}{2}$. (pp. 225–229)
3. The two triangles shown are similar. Find the missing length. (pp. A14–A19)





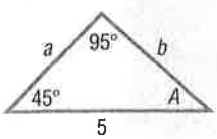
Concepts and Vocabulary

4. If none of the angles of a triangle is a right angle, the triangle is called _____.
5. For a triangle with sides a, b, c and opposite angles A, B, C, the Law of Sines states that _____.
6. ***True or False*** An oblique triangle in which two sides and an angle are given always results in at least one triangle.
7. ***True or False*** The sum of the angles of any triangle equals 180°.
8. ***True or False*** The ambiguous case refers to the fact that, when two sides and the angle opposite one of them are given, sometimes the Law of Sines cannot be used.

Skill Building

In Problems 9–16, solve each triangle.

9.



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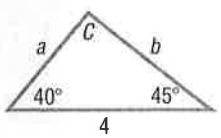

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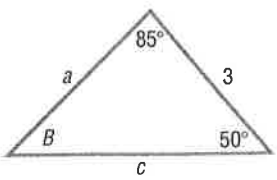
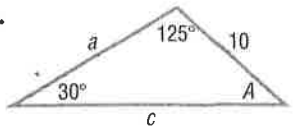


12.



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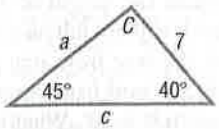

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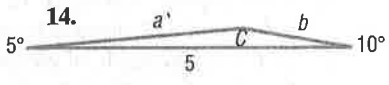

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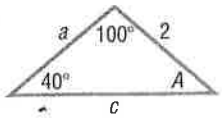
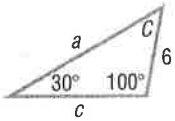


16.



4.2

In Problems 17–24, solve each triangle.

17. $A = 40°$, $B = 20°$, $a = 2$

18. $A = 50°$, $C = 20°$, $a = 3$

19. $B = 70°$, $C = 10°$, $b = 5$

20. $A = 70°$, $B = 60°$, $c = 4$

21. $A = 110°$, $C = 30°$, $c = 3$

22. $B = 10°$, $C = 100°$, $b = 2$

23. $A = 40°$, $B = 40°$, $c = 2$

24. $B = 20°$, $C = 70°$, $a = 1$

In Problems 25–36, two sides and an angle are given. Determine whether the given information results in one triangle, two triangles, or no triangle at all. Solve any triangle(s) that results.

25. $a = 3$, $b = 2$, $A = 50°$

26. $b = 4$, $c = 3$, $B = 40°$

27. $b = 5$, $c = 3$, $B = 100°$

28. $a = 2$, $c = 1$, $A = 120°$

29. $a = 4$, $b = 5$, $A = 60°$

30. $b = 2$, $c = 3$, $B = 40°$

31. $b = 4$, $c = 6$, $B = 20°$

32. $a = 3$, $b = 7$, $A = 70°$

33. $a = 2$, $c = 1$, $C = 100°$

34. $b = 4$, $c = 5$, $B = 95°$

35. $a = 2$, $c = 1$, $C = 25°$

36. $b = 4$, $c = 5$, $B = 40°$

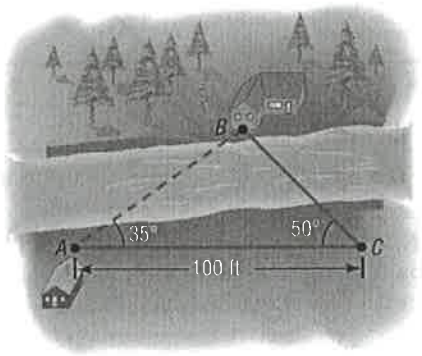
Applications and Extensions

37. Rescue at Sea Coast Guard Station Able is located 150 miles due south of Station Baker. A ship at sea sends an SOS call that is received by each station. The call to Station Able indicates that the ship is located N55°E; the call to Station Baker indicates that the ship is located S60°E.

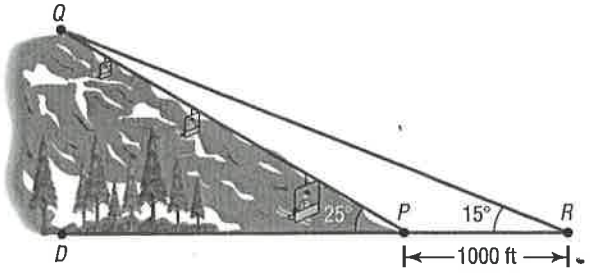
(a) How far is each station from the ship?

(b) If a helicopter capable of flying 200 miles per hour is dispatched from the station nearest the ship, how long will it take to reach the ship?

38. Surveying Consult the figure below. To find the distance from the house at A to the house at B, a surveyor measures $\angle BAC$ to be 35° and then walks off a distance of 100 feet to C and measures $\angle ACB$ to be 50°. What is the distance from A to B?

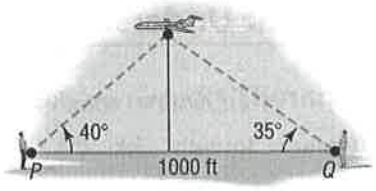


39. Finding the Length of a Ski Lift Consult the figure. To find the length of the span of a proposed ski lift from P to Q, a surveyor measures $\angle DPQ$ to be 25° and then walks off a distance of 1000 feet to R and measures $\angle PRQ$ to be 15°. What is the distance from P to Q?



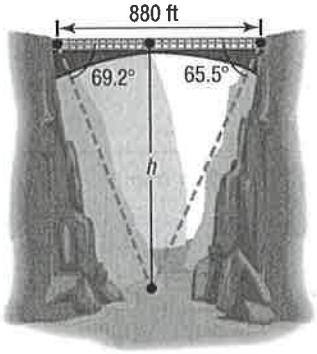
40. Finding the Height of a Mountain Use the illustration in Problem 39 to find the height QD of the mountain.

41. Finding the Height of an Airplane An aircraft is spotted by two observers who are 1000 feet apart. As the airplane passes over the line joining them, each observer takes a sighting of the angle of elevation to the plane, as indicated in the figure. How high is the airplane?



42. Finding the Height of the Bridge over the Royal Gorge The highest bridge in the world is the bridge over the Royal Gorge of the Arkansas River in Colorado. Sightings to the same point at water level directly under the bridge are taken from each side of the 880-foot-long bridge, as indicated in the figure. How high is the bridge?

Source: *Guinness Book of World Records*



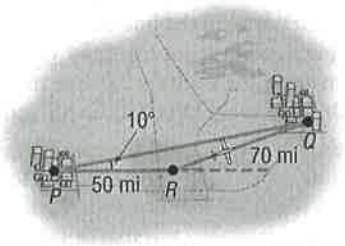
43. Landscaping Pat needs to determine the height of a tree before cutting it down to be sure that it will not fall on a nearby fence. The angle of elevation of the tree from one position on a flat path from the tree is 30°, and from a second position 40 feet farther along this path it is 20°. What is the height of the tree?

44. Construction A loading ramp 10 feet long that makes an angle of 18° with the horizontal is to be replaced by one that makes an angle of 12° with the horizontal. How long is the new ramp?

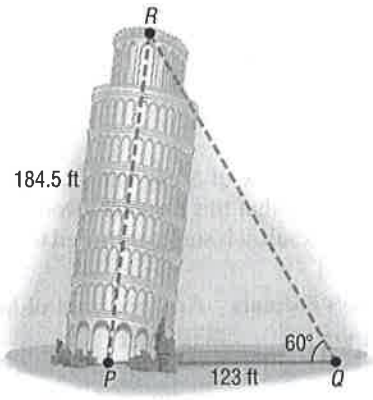
45. Commercial Navigation Adam must fly home to St. Louis from a business meeting in Oklahoma City. One flight option flies directly to St. Louis, a distance of about 461.1 miles. A second flight option flies first to Kansas City and then connects to St. Louis. The bearing from Oklahoma City to Kansas City is N29.6°E, and the bearing from Oklahoma City to St. Louis is N57.7°E. The bearing from St. Louis to Oklahoma City is S62.1°W, and the bearing from St. Louis to Kansas City is N79.4°W. How many more frequent flyer miles will Adam receive if he takes the connecting flight rather than the direct flight?

Source: www.landings.com

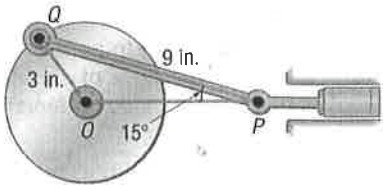
46. Time Lost due to a Navigation Error In attempting to fly from city P to city Q, an aircraft followed a course that was 10° in error, as indicated in the figure. After flying a distance of 50 miles, the pilot corrected the course by turning at point R and flying 70 miles farther. If the constant speed of the aircraft was 250 miles per hour, how much time was lost due to the error?



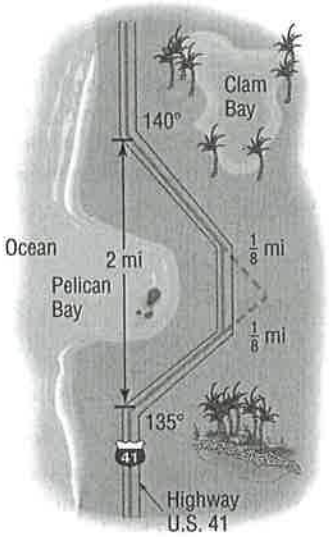
47. Finding the Lean of the Leaning Tower of Pisa The famous Leaning Tower of Pisa was originally 184.5 feet high.* At a distance of 123 feet from the base of the tower, the angle of elevation to the top of the tower is found to be 60°. Find $\angle RPQ$ indicated in the figure. Also, find the perpendicular distance from R to PQ.



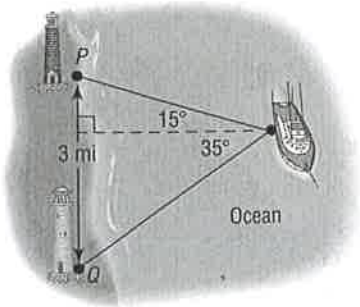
48. Crankshafts on Cars On a certain automobile, the crankshaft is 3 inches long and the connecting rod is 9 inches long (see the figure). At the time when $\angle OPQ$ is 15°, how far is the piston (P) from the center (O) of the crankshaft?



49. Constructing a Highway U.S. 41, a highway whose primary directions are north–south, is being constructed along the west coast of Florida. Near Naples, a bay obstructs the straight path of the road. Since the cost of a bridge is prohibitive, engineers decide to go around the bay. The illustration shows the path that they decide on and the measurements taken. What is the length of highway needed to go around the bay?



50. Calculating Distances at Sea The navigator of a ship at sea spots two lighthouses that she knows to be 3 miles apart along a straight seashore. She determines that the angles formed between two line-of-sight observations of the lighthouses and the line from the ship directly to shore are 15° and 35°. See the illustration.



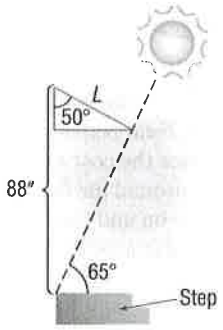
* On February 27, 1964, the government of Italy requested aid in preventing the tower from toppling. A multinational task force of engineers, mathematicians, and historians was assigned and met on the Azores islands to discuss stabilization methods. After over two decades of work on the subject, the tower was closed to the public in January 1990. During the time that the tower was closed, the bells were removed to relieve some weight and cables were cinched around the third level and anchored several hundred meters away. Apartments and houses in the path of the tower were vacated for safety concerns. After a decade of corrective reconstruction and stabilization efforts, the tower was reopened to the public on December 15, 2001. Many methods were proposed to stabilize the tower, including the addition of 800 metric tons of lead counterweights to the raised end of the base. The final solution to correcting the lean was to remove 38 cubic meters of soil from underneath the raised end. The tower has been declared stable for at least another 300 years.

Source: http://en.wikipedia.org/wiki/Leaning_Tower_of_Pisa, page last modified June 28, 2006

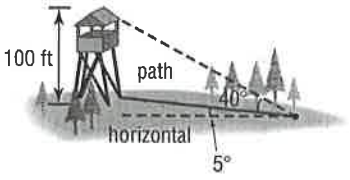
4.2

(a) How far is the ship from lighthouse P?
(b) How far is the ship from lighthouse Q?
(c) How far is the ship from shore?

51. Designing an Awning An awning that covers a sliding glass door that is 88 inches tall forms an angle of 50° with the wall. The purpose of the awning is to prevent sunlight from entering the house when the angle of elevation of the Sun is more than 65°. See the figure. Find the length L of the awning.

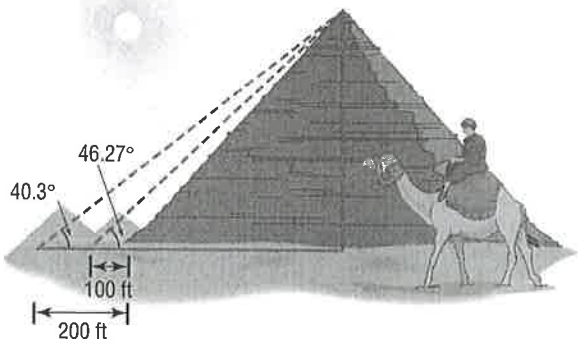


52. Finding Distances A forest ranger is walking on a path inclined at 5° to the horizontal directly toward a 100-foot-tall fire observation tower. The angle of elevation from the path to the top of the tower is 40°. How far is the ranger from the tower at this time?



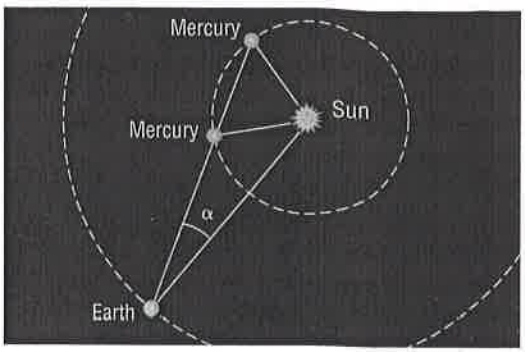
53. Great Pyramid of Cheops One of the original Seven Wonders of the World, the Great Pyramid of Cheops was built about 2580 BC. Its original height was 480 feet 11 inches, but due to the loss of its topmost stones, it is now shorter. Find the current height of the Great Pyramid, using the information given in the illustration.

Source: *Guinness Book of World Records*



54. Determining the Height of an Aircraft Two sensors are spaced 700 feet apart along the approach to a small airport. When an aircraft is nearing the airport, the angle of elevation from the first sensor to the aircraft is 20°, and from the second sensor to the aircraft it is 15°. Determine how high the aircraft is at this time.

55. Mercury The distance from the Sun to Earth is approximately 149,600,000 kilometers (km). The distance from Sun to Mercury is approximately 57,910,000 km. The **elongation angle** α is the angle formed between the line of sight from Earth to the Sun and the line of sight from Earth Mercury. See the figure. Suppose that the elongation an for Mercury is 15°. Use this information to find the possi distances between Earth and Mercury.



56. Venus The distance from the Sun to Earth is approximat 149,600,000 km. The distance from the Sun to Venus is a proximately 108,200,000 km. The elongation angle α is t angle formed between the line of sight from Earth to the S and the line of sight from Earth to Venus. Suppose that t elongation angle for Venus is 10°. Use this information to fi the possible distances between Earth and Venus.

57. The Original Ferris Wheel George Washington Gale Fer Jr. designed the original Ferris wheel for the 1893 Worl Columbian Exposition in Chicago, Illinois. The wheel had equally spaced cars each the size of a school bus. The distan between adjacent cars was approximately 22 feet. Determi the diameter of the wheel to the nearest foot.

Source: *Carnegie Library of Pittsburgh, www.clpgh.org*

58. Mollweide's Formula For any triangle, Mollweide's Formu (named after Karl Mollweide, 1774–1825) states that

$$\frac{a+b}{c} = \frac{\cos\left[\frac{1}{2}(A-B)\right]}{\sin\left(\frac{1}{2}C\right)}$$

Derive it.

[**Hint:** Use the Law of Sines and then a Sum-to-Produ Formula. Notice that this formula involves all six parts of triangle. As a result, it is sometimes used to check the solutio of a triangle.]

59. Mollweide's Formula Another form of Mollweide's Fo mula is

$$\frac{a-b}{c} = \frac{\sin\left[\frac{1}{2}(A-B)\right]}{\cos\left(\frac{1}{2}C\right)}$$

Derive it.

60. For any triangle, derive the formula

$$a = b\cos C + c\cos B$$

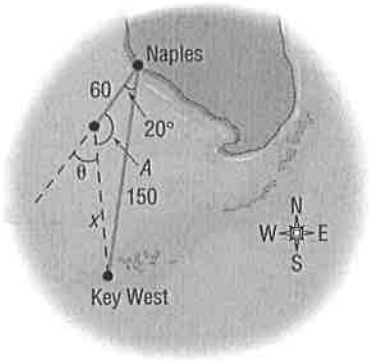
[**Hint:** Use the fact that $\sin A = \sin(180° - B - C)$.]

4.3

(a) How far is the sailboat from Key West at this time?
(b) Through what angle should the sailboat turn to correct its course?
(c) How much time has been added to the trip because of this? (Assume th speed remains at 15 miles per hour.)

Solution See Figure 38. With a speed of 15 miles per hour, the sailboat has gone 60 after 4 hours. We seek the distance x of the sailboat from Key West. We als the angle θ that the sailboat should turn through to correct its course.

Figure 38



(a) To find x, we use the Law of Cosines, since we know two sides and the in angle.

$$x^2 = 150^2 + 60^2 - 2(150)(60)\cos 20° \approx 9186.53$$
$$x \approx 95.8$$

The sailboat is about 96 miles from Key West.

(b) We now know three sides of the triangle, so we can use the Law of C again to find the angle A opposite the side of length 150 miles.

$$150^2 = 96^2 + 60^2 - 2(96)(60)\cos A$$
$$9684 = -11{,}520\cos A$$
$$\cos A \approx -0.8406$$
$$A \approx 147.2°$$

The sailboat should turn through an angle of

$$\theta = 180° - A \approx 180° - 147.2° = 32.8°$$

The sailboat should turn through an angle of about 33° to correct its c

(c) The total length of the trip is now 60 + 96 = 156 miles. The extra 6 mil only require about 0.4 hour or 24 minutes more if the speed of 15 mil hour is maintained.

Now Work PROBLEM 35

Historical Feature

The Law of Sines was known vaguely long before it was explicitly stated by Nasir Eddin (about AD 1250). Ptolemy (about AD 150) was aware of it in a form using a chord function instead of the sine function. But it was first clearly stated in Europe by Regiomontanus, writing in 1464.

The Law of Cosines appears first in Euclid's *Elements* (Book II), but in a well-disguised form in which squares built on the sides of triangles are added and a rectangle representing the cosine term is subtracted. It was thus known to all mathematicians because of their familiarity with Euclid's work. An early modern form of the Law of Cosines, that ing the angle when the sides are known, was stated by Franç (in 1593).

The Law of Tangents (see Problem 61 of Exercise 4.2) has obsolete. In the past it was used in place of the Law of Cosines, the Law of Cosines was very inconvenient for calculation with log or slide rules. Mixing of addition and multiplication is now very a calculator, however, and the Law of Tangents has been shelve with the slide rule.

4.3 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in*

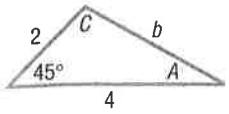
1. Write the formula for the distance d from $P_1 = (x_1, y_1)$ to $P_2 = (x_2, y_2)$. (p. 3)
2. If θ is an acute angle, solve the equation $\cos\theta$ (pp. 225–229)

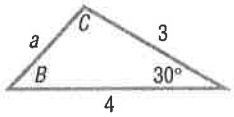
Concepts and Vocabulary

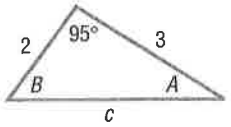
3. If three sides of a triangle are given, the Law of _____ is used to solve the triangle.
4. If one side and two angles of a triangle are given, the Law of _____ is used to solve the triangle.
5. If two sides and the included angle of a triangle are given, the Law of _____ is used to solve the triangle.
6. ***True or False*** Given only the three sides of a triangle, there is insufficient information to solve the triangle.
7. ***True or False*** Given two sides and the included angle, the first thing to do to solve the triangle is to use the Law of Sines.
8. ***True or False*** A special case of the Law of Cosines is the Pythagorean Theorem.

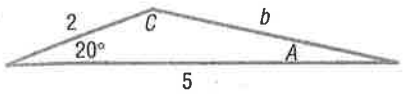
Skill Building

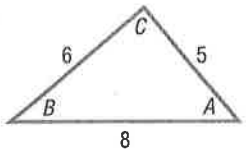
In Problems 9–16, solve each triangle.

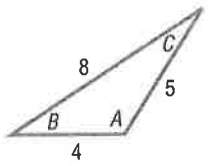
9. 

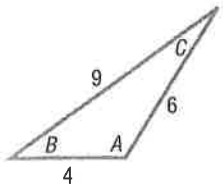
10. 

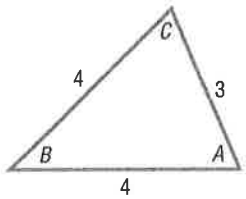
11. 

12. 

13. 

14. 

15. 

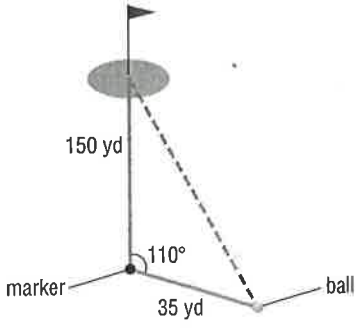
16. 

In Problems 17–32, solve each triangle.

17. $a = 3, \quad b = 4, \quad C = 40°$
18. $a = 2, \quad c = 1, \quad B = 10°$
19. $b = 1, \quad c = 3, \quad A = 80°$
20. $a = 6, \quad b = 4, \quad C = 60°$
21. $a = 3, \quad c = 2, \quad B = 110°$
22. $b = 4, \quad c = 1, \quad A = 120°$
23. $a = 2, \quad b = 2, \quad C = 50°$
24. $a = 3, \quad c = 2, \quad B = 90°$
25. $a = 12, \quad b = 13, \quad c = 5$
26. $a = 4, \quad b = 5, \quad c = 3$
27. $a = 2, \quad b = 2, \quad c = 2$
28. $a = 3, \quad b = 3, \quad c = 2$
29. $a = 5, \quad b = 8, \quad c = 9$
30. $a = 4, \quad b = 3, \quad c = 6$
31. $a = 10, \quad b = 8, \quad c = 5$
32. $a = 9, \quad b = 7, \quad c = 10$

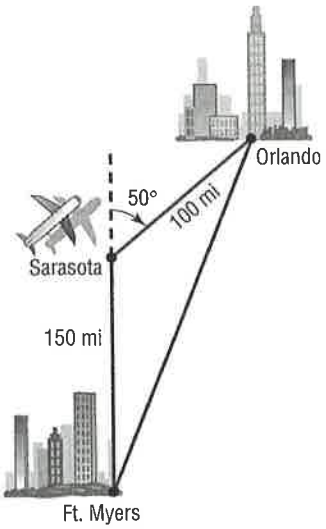
Applications and Extensions

33. **Distance to the Green** A golfer hits an errant tee shot that lands in the rough. A marker in the center of the fairway is 150 yards from the center of the green. While standing on the marker and facing the green, the golfer turns 110° toward his ball. He then paces off 35 yards to his ball. See the figure. How far is the ball from the center of the green?



34. **Navigation** An airplane flies due north from Ft. Myers to Sarasota, a distance of 150 miles, and then turns through an angle of 50° and flies to Orlando, a distance of 100 miles. See the figure on next page.

4.3



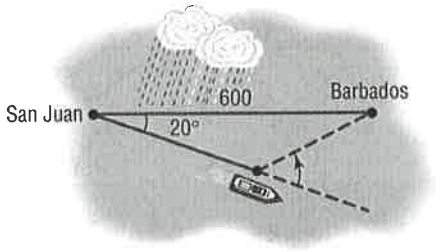
(a) How far is it directly from Ft. Myers to Orlando?

(b) What bearing should the pilot use to fly directly from Ft. Myers to Orlando?

35. Avoiding a Tropical Storm A cruise ship maintains an average speed of 15 knots in going from San Juan, Puerto Rico, to Barbados, West Indies, a distance of 600 nautical miles. To avoid a tropical storm, the captain heads out of San Juan in a direction of 20° off a direct heading to Barbados. The captain maintains the 15-knot speed for 10 hours, after which time the path to Barbados becomes clear of storms.

(a) Through what angle should the captain turn to head directly to Barbados?

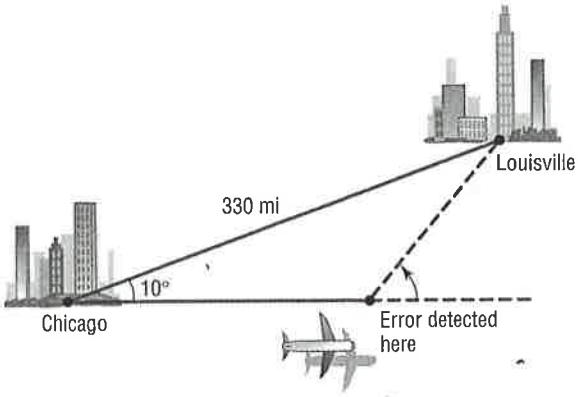
(b) Once the turn is made, how long will it be before the ship reaches Barbados if the same 15-knot speed is maintained?



36. Revising a Flight Plan In attempting to fly from Chicago to Louisville, a distance of 330 miles, a pilot inadvertently took a course that was 10° in error, as indicated in the figure.

(a) If the aircraft maintains an average speed of 220 miles per hour and if the error in direction is discovered after 15 minutes, through what angle should the pilot turn head toward Louisville?

(b) What new average speed should the pilot maintain that the total time of the trip is 90 minutes?



37. Major League Baseball Field A Major League baseball diamond is actually a square 90 feet on a side. The pitching rubber is located 60.5 feet from home plate on a line joining home plate and second base.

(a) How far is it from the pitching rubber to first base?

(b) How far is it from the pitching rubber to second base?

(c) If a pitcher faces home plate, through what angle does he need to turn to face first base?

38. Little League Baseball Field According to Little League baseball official regulations, the diamond is a square 60 feet on a side. The pitching rubber is located 46 feet from home plate on a line joining home plate and second base.

(a) How far is it from the pitching rubber to first base?

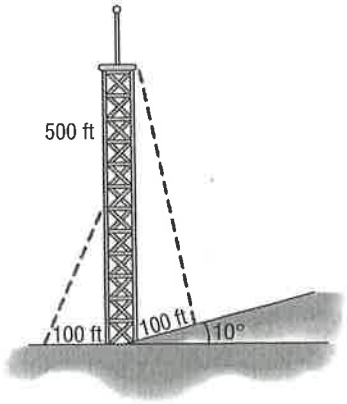
(b) How far is it from the pitching rubber to second base?

(c) If a pitcher faces home plate, through what angle does he need to turn to face first base?

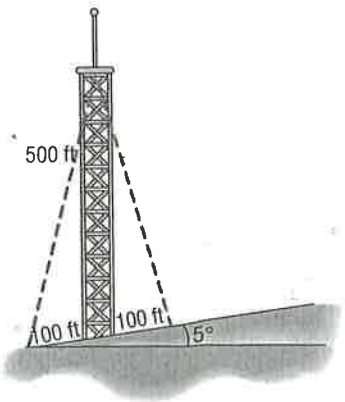
39. Finding the Length of a Guy Wire The height of a radio tower is 500 feet, and the ground on one side of the tower slopes upward at an angle of 10° (see the figure).

(a) How long should a guy wire be if it is to connect the top of the tower and be secured at a point on the sloping side 100 feet from the base of the tower?

(b) How long should a second guy wire be if it is to connect to the middle of the tower and be secured at a point 100 feet from the base on the flat side?

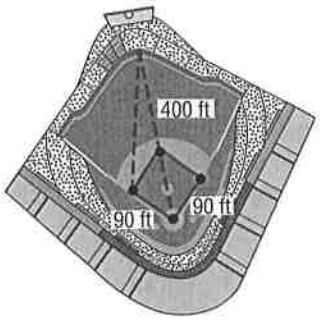


40. Finding the Length of a Guy Wire See the figure below. A radio tower 500 feet high is located on the side of a hill



an inclination to the horizontal of 5°. How long should two guy wires be if they are to connect to the top of the tower and be secured at two points 100 feet directly above and directly below the base of the tower?

41. Wrigley Field, Home of the Chicago Cubs The distance from home plate to the fence in dead center in Wrigley Field is 400 feet (see the figure). How far is it from the fence in dead center to third base?



42. Little League Baseball The distance from home plate to the fence in dead center at the Oak Lawn Little League field is 280 feet. How far is it from the fence in dead center to third base?

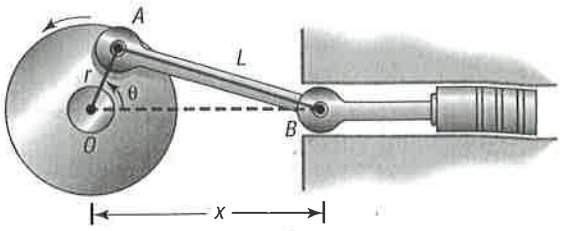
[**Hint:** The distance between the bases in Little League is 60 feet.]

43. Building a Swing Set Clint is building a wooden swing set for his children. Each supporting end of the swing set is to be an A-frame constructed with two 10-foot-long 4 by 4s joined at a 45° angle. To prevent the swing set from tipping over, Clint wants to secure the base of each A-frame to concrete footings. How far apart should the footings for each A-frame be?

44. Rods and Pistons Rod OA rotates about the fixed point O so that point A travels on a circle of radius r. Connected to point A is another rod AB of length $L > 2r$, and point B is connected to a piston. See the figure. Show that the distance x between point O and point B is given by

$$x = r\cos\theta + \sqrt{r^2\cos^2\theta + L^2 - r^2}$$

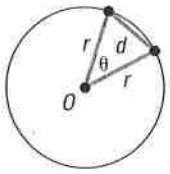
where θ is the angle of rotation of rod OA.



45. Geometry Show that the length d of a chord of a circle of radius r is given by the formula

$$d = 2r\sin\frac{\theta}{2}$$

where θ is the central angle formed by the radii to the ends of the chord. See the figure. Use this result to derive the fact that $\sin\theta < \theta$, where $\theta > 0$ is measured in radians.



46. For any triangle, show that

$$\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$$

where $s = \frac{1}{2}(a + b + c)$.

[**Hint:** Use a Half-angle Formula and the Law of Cosines.]

47. For any triangle show that

$$\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$$

where $s = \frac{1}{2}(a + b + c)$.

48. Use the Law of Cosines to prove the identity

$$\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2 + b^2 + c^2}{2abc}$$

Discussion and Writing

49. What do you do first if you are asked to solve a triangle and are given two sides and the included angle?

50. What do you do first if you are asked to solve a triangle and are given three sides?

51. Make up an applied problem that requires using the Law of Cosines.

52. Write down your strategy for solving an oblique triangle.

'Are You Prepared?' Answers

1. $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ **2.** $\theta = 45°$

4.4 Area of a Triangle

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Geometry Essentials (Appendix A, Section A.2, pp. A14–A19)

Now Work the 'Are You Prepared?' problem on page 278.

OBJECTIVES 1 Find the Area of SAS Triangles (p. 276)
2 Find the Area of SSS Triangles (p. 277)

4.4

Similarly, using $\sin^2\frac{C}{2} = \frac{1-\cos C}{2}$, we find that

$$\sin^2\frac{C}{2} = \frac{(s-a)(s-b)}{ab} \tag{7}$$

Now we use formula (2) for the area.

$$\begin{aligned} K &= \frac{1}{2}ab\sin C \\ &= \frac{1}{2}ab \cdot 2\sin\frac{C}{2}\cos\frac{C}{2} && \sin C = \sin\left[2\left(\frac{C}{2}\right)\right] = 2\sin\frac{C}{2}\cos\frac{C}{2} \\ &= ab\sqrt{\frac{(s-a)(s-b)}{ab}}\sqrt{\frac{s(s-c)}{ab}} && \text{Use equations (6) and (7).} \\ &= \sqrt{s(s-a)(s-b)(s-c)} \end{aligned}$$

Historical Feature

Heron's Formula (also known as *Hero's Formula*) is due to Heron of Alexandria (first century AD), who had, besides his mathematical talents, a good deal of engineering skills. In various temples his mechanical devices produced effects that seemed supernatural, and visitors presumably were thus influenced to generosity. Heron's book *Metrica*, on making such devices, has survived and was discovered in 1896 in the city of Constantinople.

Heron's Formulas for the area of a triangle caused some n comfort in Greek mathematics, because a product with two fact an area, while one with three factors was a volume, but four seemed contradictory in Heron's time.

4.4 Assess Your Understanding

'Are You Prepared?' *Answer as given at the end of these exercises. If you get a wrong answer, read the page listed in red.*

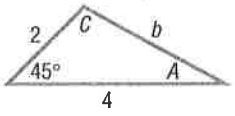
1. The area of a triangle whose base is b and whose height is h is _____ (pp. A14–A19)

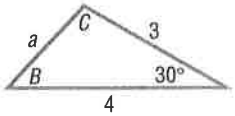
Concepts and Vocabulary

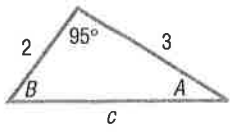
2. If three sides of a triangle are given, _____ Formula is used to find the area of the triangle.
3. ***True or False*** No formula exists for finding the area of a triangle when only three sides are given.
4. ***True or False*** Given two sides and the included angl is a formula that can be used to find the area of the t

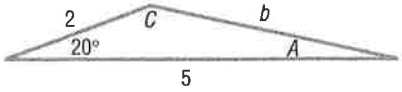
Skill Building

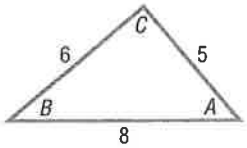
In Problems 5–12, find the area of each triangle. Round answers to two decimal places.

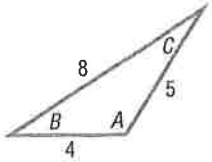
5. 

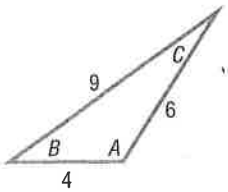
6. 

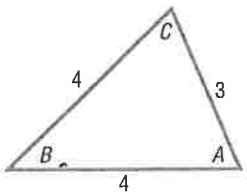
7. 

8. 

9. 

10. 

11. 

12. 

In Problems 13–24, find the area of each triangle. Round answers to two decimal places.

13. $a = 3,\ b = 4,\ C = 40°$ **14.** $a = 2,\ c = 1,\ B = 10°$ **15.** $b = 1,\ c = 3,\ A = 80°$

16. $a = 6,\ b = 4,\ C = 60°$ **17.** $a = 3,\ c = 2,\ B = 110°$ **18.** $b = 4,\ c = 1,\ A = 120°$

19. $a = 12,\ b = 13,\ c = 5$ **20.** $a = 4,\ b = 5,\ c = 3$ **21.** $a = 2,\ b = 2,\ c = 2$

22. $a = 3,\ b = 3,\ c = 2$ **23.** $a = 5,\ b = 8,\ c = 9$ **24.** $a = 4,\ b = 3,\ c = 6$

Applications and Extensions

25. Area of an ASA Triangle If two angles and the included side are given, the third angle is easy to find. Use the Law of Sines to show that the area K of a triangle with side a and angles A, B, and C is

$$K = \frac{a^2 \sin B \sin C}{2 \sin A}$$

26. Area of a Triangle Prove the two other forms of the formula given in Problem 25.

$$K = \frac{b^2 \sin A \sin C}{2 \sin B} \quad \text{and} \quad K = \frac{c^2 \sin A \sin B}{2 \sin C}$$

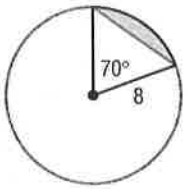
In Problems 27–32, use the results of Problem 25 or 26 to find the area of each triangle. Round answers to two decimal places.

27. $A = 40°,\ B = 20°,\ a = 2$ **28.** $A = 50°,\ C = 20°,\ a = 3$ **29.** $B = 70°,\ C = 10°,\ b = 5$

30. $A = 70°,\ B = 60°,\ c = 4$ **31.** $A = 110°,\ C = 30°,\ c = 3$ **32.** $B = 10°,\ C = 100°,\ b = 2$

33. Area of a Segment Find the area of the segment (shaded in blue in the figure) of a circle whose radius is 8 feet, formed by a central angle of 70°.

[**Hint:** Subtract the area of the triangle from the area of the sector to obtain the area of the segment.]

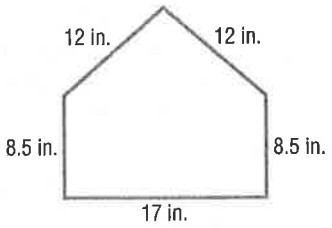


34. Area of a Segment Find the area of the segment of a circle whose radius is 5 inches, formed by a central angle of 40°.

35. Cost of a Triangular Lot The dimensions of a triangular lot are 100 feet by 50 feet by 75 feet. If the price of such land is \$3 per square foot, how much does the lot cost?

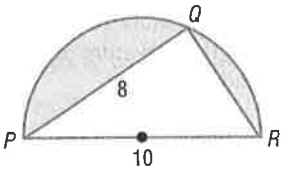
36. Amount of Material to Make a Tent A cone-shaped tent is made from a circular piece of canvas 24 feet in diameter by removing a sector with central angle 100° and connecting the ends. What is the surface area of the tent?

37. Dimensions of Home Plate The dimensions of home plate at any major league baseball stadium are shown. Find the area of home plate.

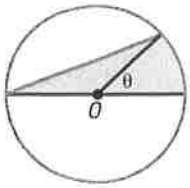


38. Computing Areas Find the area of the shaded region enclosed in a semicircle of diameter 10 inches. The length of the chord PQ is 8 inches.

[**Hint:** Triangle PQR is a right triangle.]

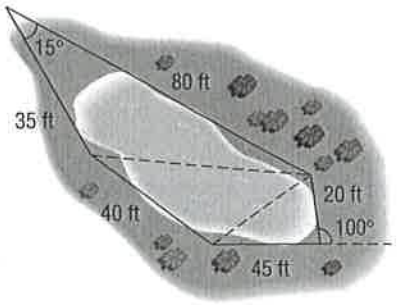


39. Geometry Consult the figure, which shows a circle of radius r with center at O. Find the area K of the shaded region as a function of the central angle θ.



40. Approximating the Area of a Lake To approximate the area of a lake, a surveyor walks around the perimeter of the lake, taking the measurements shown in the illustration. Using this technique, what is the approximate area of the lake?

[**Hint:** Use the Law of Cosines on the three triangles shown and then find the sum of their areas.]



41. The Flatiron Building Completed in 1902 in New York City, the Flatiron Building is triangular shaped and bounded by 22nd Street, Broadway, and 5th Avenue. The building measures approximately 87 feet on the 22nd Street side, 190 feet on the Broadway side, and 173 feet on the 5th Avenue side. Approximate the ground area covered by the building.