Chapter 3

The domain of $f(x) = 2 \sin x - 1$ is the set of all real numbers. To find the domain of f^{-1}, we note that the argument of the inverse sine function is $\frac{x+1}{2}$ and that it must lie in the interval $[-1, 1]$. That is,

$$-1 \le \frac{x+1}{2} \le 1$$
$$-2 \le x + 1 \le 2 \quad \text{Multiply each part by 2.}$$
$$-3 \le x \le 1 \quad \text{Add } -1 \text{ to each part.}$$

The domain of f^{-1} is $\{x \mid -3 \le x \le 1\}$.

Now Work PROBLEM 55

5 Solve Equations Involving Inverse Trigonometric Functions

Equations that contain inverse trigonometric functions are called **inverse trigonometric equations.**

EXAMPLE 11 Solving an Equation Involving an Inverse Trigonometric Function

Solve the equation: $3 \sin^{-1} x = \pi$

Solution To solve an equation involving a single inverse trigonometric function, first solve the equation for the inverse trigonometric function.

$$3 \sin^{-1} x = \pi$$
$$\sin^{-1} x = \frac{\pi}{3} \quad \text{Divide both sides by 3.}$$
$$x = \sin \frac{\pi}{3} \quad y = \sin^{-1} x \text{ means } x = \sin y.$$
$$x = \frac{\sqrt{3}}{2}$$

The solution set is $\left\{\frac{\sqrt{3}}{2}\right\}$.

Now Work PROBLEM 61

3.1 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. What is the domain and the range of $y = \sin x$? (pp. 123–127)
2. A suitable restriction on the domain of the function $f(x) = (x - 1)^2$ to make it one-to-one would be _____. (pp. 71–80)
3. If the domain of a one-to-one function is $[3, \infty)$, the range of its inverse is _____. (pp. 71–80)
4. ***True or False*** The graph of $y = \cos x$ is decreasing on the interval $[0, \pi]$. (p. 139)
5. $\tan \frac{\pi}{4} =$ _____; $\sin \frac{\pi}{3} =$ _____ (p. 115)
6. $\sin\left(-\frac{\pi}{6}\right) =$ _____; $\cos \pi =$ _____ (pp. 112, 115, and 123–127)

Concepts and Vocabulary

7. $y = \sin^{-1} x$ means _____, where $-1 \le x \le 1$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.
8. The value of $\sin^{-1}\left[\sin \frac{\pi}{2}\right]$ is _____.

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9. $\cos^{-1}\left[\cos\frac{\pi}{5}\right] =$ ______.

10. ***True or False*** The domain of $y = \sin^{-1} x$ is $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.

11. ***True or False*** $\sin(\sin^{-1} 0) = 0$ and $\cos(\cos^{-1} 0) = 0$.

12. ***True or False*** $y = \tan^{-1} x$ means $x = \tan y$, where $-\infty < x < \infty$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

Skill Building

In Problems 13–24, find the exact value of each expression.

13. $\sin^{-1} 0$
14. $\cos^{-1} 1$
15. $\sin^{-1}(-1)$
16. $\cos^{-1}(-1)$
17. $\tan^{-1} 0$
18. $\tan^{-1}(-1)$
19. $\sin^{-1}\frac{\sqrt{2}}{2}$
20. $\tan^{-1}\frac{\sqrt{3}}{3}$
21. $\tan^{-1}\sqrt{3}$
22. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
23. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
24. $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

In Problems 25–36, use a calculator to find the value of each expression rounded to two decimal places.

25. $\sin^{-1} 0.1$
26. $\cos^{-1} 0.6$
27. $\tan^{-1} 5$
28. $\tan^{-1} 0.2$
29. $\cos^{-1}\frac{7}{8}$
30. $\sin^{-1}\frac{1}{8}$
31. $\tan^{-1}(-0.4)$
32. $\tan^{-1}(-3)$
33. $\sin^{-1}(-0.12)$
34. $\cos^{-1}(-0.44)$
35. $\cos^{-1}\frac{\sqrt{2}}{3}$
36. $\sin^{-1}\frac{\sqrt{3}}{5}$

In Problems 37–44, find the exact value of each expression. Do not use a calculator.

37. $\cos^{-1}\left(\cos\frac{4\pi}{5}\right)$
38. $\sin^{-1}\left[\sin\left(-\frac{\pi}{10}\right)\right]$
39. $\tan^{-1}\left[\tan\left(-\frac{3\pi}{8}\right)\right]$
40. $\sin^{-1}\left[\sin\left(-\frac{3\pi}{7}\right)\right]$
41. $\sin^{-1}\left(\sin\frac{9\pi}{8}\right)$
42. $\cos^{-1}\left[\cos\left(-\frac{5\pi}{3}\right)\right]$
43. $\tan^{-1}\left(\tan\frac{4\pi}{5}\right)$
44. $\tan^{-1}\left[\tan\left(-\frac{2\pi}{3}\right)\right]$

In Problems 45–52, find the exact value, if any, of each composite function. If there is no value, say it is "not defined." Do not use calculator.

45. $\sin\left(\sin^{-1}\frac{1}{4}\right)$
46. $\cos\left[\cos^{-1}\left(-\frac{2}{3}\right)\right]$
47. $\tan(\tan^{-1} 4)$
48. $\tan[\tan^{-1}(-2)]$
49. $\cos(\cos^{-1} 1.2)$
50. $\sin[\sin^{-1}(-2)]$
51. $\tan(\tan^{-1}\pi)$
52. $\sin[\sin^{-1}(-1.5)]$

In Problems 53–60, find the inverse function f^{-1} of each function f. State the domain of f and f^{-1}.

53. $f(x) = 5\sin x + 2$
54. $f(x) = 2\tan x - 3$
55. $f(x) = -2\cos(3x)$
56. $f(x) = 3\sin(2x)$
57. $f(x) = -\tan(x + 1) - 3$
58. $f(x) = \cos(x + 2) + 1$
59. $f(x) = 3\sin(2x + 1)$
60. $f(x) = 2\cos(3x + 2)$

In Problems 61–68, find the exact solution of each equation.

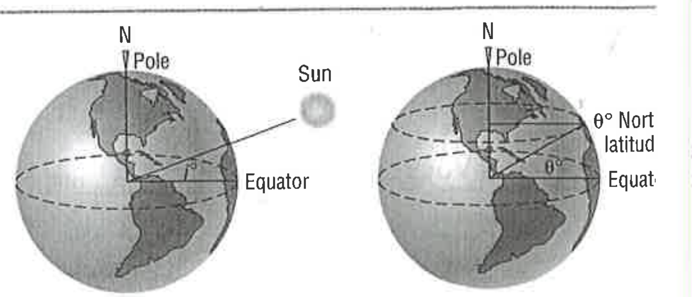
61. $4\sin^{-1}(x) = \pi$
62. $2\cos^{-1} x = \pi$
63. $3\cos^{-1}(2x) = 2\pi$
64. $-6\sin^{-1}(3x) = \pi$
65. $3\tan^{-1} x = \pi$
66. $-4\tan^{-1} x = \pi$
67. $4\cos^{-1} x - 2\pi = 2\cos^{-1} x$
68. $5\sin^{-1} x - 2\pi = 2\sin^{-1} x - 3\pi$

Applications and Extensions

In Problems 69–74, use the following discussion. The formula

$$D = 24\left[1 - \frac{\cos^{-1}(\tan i \tan\theta)}{\pi}\right]$$

can be used to approximate the number of hours of daylight D when the declination of the Sun is $i°$ at a location $\theta°$ north latitude for any date between the vernal equinox and autumnal equinox. The declination of the Sun is defined as the angle i between the equatorial plane and any ray of light from the Sun. The latitude of a location is



3.1

the angle θ between the Equator and the location on the surface of Earth, with the vertex of the angle located at the center of Earth. See the figure. To use the formula, $\cos^{-1}(\tan i \tan \theta)$ *must be expressed in radians.*

69. Approximate the number of hours of daylight in Houston, Texas (29°45′ north latitude), for the following dates:
(a) Summer solstice ($i = 23.5°$)
(b) Vernal equinox ($i = 0°$)
(c) July 4 ($i = 22°48'$)

70. Approximate the number of hours of daylight in New York, New York (40°45′ north latitude), for the following dates:
(a) Summer solstice ($i = 23.5°$)
(b) Vernal equinox ($i = 0°$)
(c) July 4 ($i = 22°48'$)

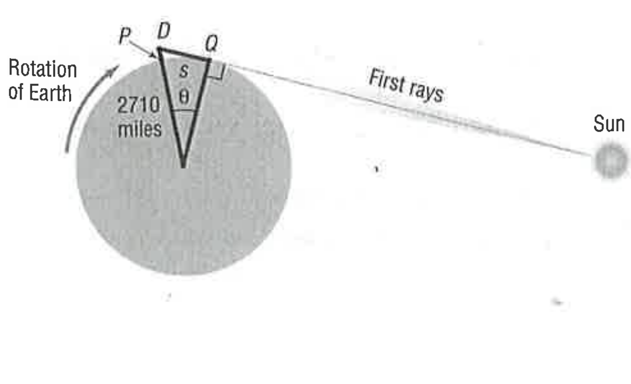
71. Approximate the number of hours of daylight in Honolulu, Hawaii (21°18′ north latitude), for the following dates:
(a) Summer solstice ($i = 23.5°$)
(b) Vernal equinox ($i = 0°$)
(c) July 4 ($i = 22°48'$)

72. Approximate the number of hours of daylight in Anchorage, Alaska (61°10′ north latitude), for the following dates:
(a) Summer solstice ($i = 23.5°$)
(b) Vernal equinox ($i = 0°$)
(c) July 4 ($i = 22°48'$)

73. Approximate the number of hours of daylight at the Equator (0° north latitude) for the following dates:
(a) Summer solstice ($i = 23.5°$)
(b) Vernal equinox ($i = 0°$)
(c) July 4 ($i = 22°48'$)
(d) What do you conclude about the number of hours of daylight throughout the year for a location at the Equator?

74. Approximate the number of hours of daylight for any location that is 66°30′ north latitude for the following dates:
(a) Summer solstice ($i = 23.5°$)
(b) Vernal equinox ($i = 0°$)
(c) July 4 ($i = 22°48'$)
(d) The number of hours of daylight on the winter solstice may be found by computing the number of hours of daylight on the summer solstice and subtracting this result from 24 hours, due to the symmetry of the orbital path of Earth around the Sun. Compute the number of hours of daylight for this location on the winter solstice. What do you conclude about daylight for a location at 66°30′ north latitude?

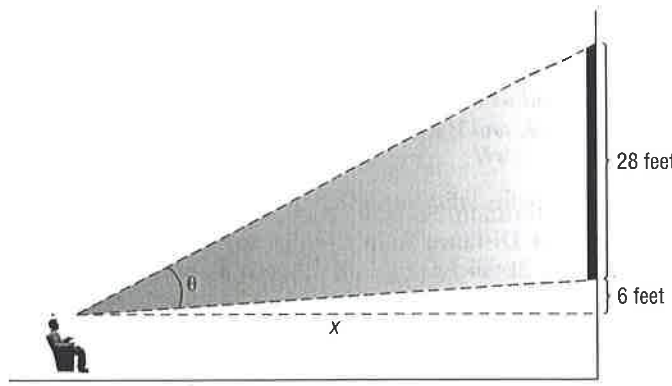
75. Being the First to See the Rising Sun Cadillac Mountain, elevation 1530 feet, is located in Acadia National Park, Maine, and is the highest peak on the east coast of the United States. It is said that a person standing on the summit will be the first person in the United States to see the rays of the rising Sun.



How much sooner would a person atop Cadillac Mountain see the first rays than a person standing below, at sea level? [**Hint:** Consult the figure. When the person at D sees the first rays of the Sun, the person at P does not. The person at P sees the first rays of the Sun only after Earth has rotated so that P is at location Q. Compute the length of the arc subtended by the central angle θ. Then use the fact that, at the latitude of Cadillac Mountain, in 24 hours a length of $2\pi(2710) \approx 17027.4$ miles is subtended, and find the time that it takes to subtend this length.]

76. Movie Theater Screens Suppose that a movie theater has a screen that is 28 feet tall. When you sit down, the bottom of the screen is 6 feet above your eye level. The angle formed by drawing a line from your eye to the bottom of the screen and your eye and the top of the screen is called the **viewing angle.** In the figure θ is the viewing angle. Suppose that you sit x feet from the screen. The viewing angle θ is given by the function

$$\theta(x) = \tan^{-1}\left(\frac{34}{x}\right) - \tan^{-1}\left(\frac{6}{x}\right).$$



(a) What is your viewing angle if you sit 10 feet from the screen? 15 feet? 20 feet?
(b) If there is 5 feet between the screen and the first row of seats and there is 3 feet between each row, which row results in the largest viewing angle?
(c) Using a graphing utility, graph

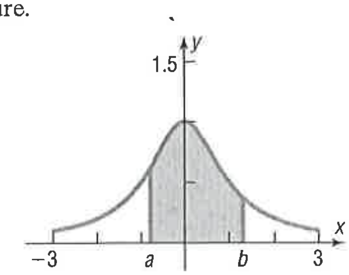
$$\theta(x) = \tan^{-1}\left(\frac{34}{x}\right) - \tan^{-1}\left(\frac{6}{x}\right)$$

What value of x results in the largest viewing angle?

77. Area under a Curve The area under the graph of $y = \dfrac{1}{1 + x^2}$ and above the x-axis between $x = a$ and $x = b$ is given by

$$\tan^{-1} b - \tan^{-1} a$$

See the figure.



3.2

Solution Let $\theta = \tan^{-1}u$ so that $\tan\theta = u, -\frac{\pi}{2} < \theta < \frac{\pi}{2}, -\infty < u < \infty$. As a result, w know that $\sec\theta > 0$. Then

$$\sin(\tan^{-1}u) = \sin\theta = \sin\theta \cdot \frac{\cos\theta}{\cos\theta} = \tan\theta\cos\theta = \frac{\tan\theta}{\sec\theta} = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}} = \frac{u}{\sqrt{1+u^2}}$$

Multiply by 1: $\frac{\cos\theta}{\cos\theta}$; $\frac{\sin\theta}{\cos\theta} = \tan\theta$; $\sec^2\theta = 1 + \tan^2\theta$, $\sec\theta > 0$

Now Work PROBLEM 57

3.2 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. What is the domain and the range of $y = \sec x$? (pp. 123–125)
2. ***True or False*** The graph of $y = \sec x$ is increasing on the interval $\left[0, \frac{\pi}{2}\right)$ and on the interval $\left(\frac{\pi}{2}, \pi\right]$. (pp. 154–156)
3. If $\cot\theta = -2$ and $0 < \theta < \pi$, then $\cos\theta =$ ______ (pp. 130–132)

Concepts and Vocabulary

4. $y = \sec^{-1}x$ means _____, where $|x|$ _____ and _____ $\leq y \leq$ _____, $y \neq \frac{\pi}{2}$.
5. $\cos(\tan^{-1}1) =$ ____.
6. ***True or False*** It is impossible to obtain exact values for the inverse secant function.
7. ***True or False*** $\csc^{-1}0.5$ is not defined.
8. ***True or False*** The domain of the inverse cotangent function is the set of real numbers.

Skill Building

In Problems 9–36, find the exact value of each expression.

9. $\cos\left(\sin^{-1}\frac{\sqrt{2}}{2}\right)$
10. $\sin\left(\cos^{-1}\frac{1}{2}\right)$
11. $\tan\left[\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right]$
12. $\tan\left[\sin^{-1}\left(-\frac{1}{2}\right)\right]$
13. $\sec\left(\cos^{-1}\frac{1}{2}\right)$
14. $\cot\left[\sin^{-1}\left(-\frac{1}{2}\right)\right]$
15. $\csc(\tan^{-1}1)$
16. $\sec(\tan^{-1}\sqrt{3})$
17. $\sin[\tan^{-1}(-1)]$
18. $\cos\left[\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right]$
19. $\sec\left[\sin^{-1}\left(-\frac{1}{2}\right)\right]$
20. $\csc\left[\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right]$
21. $\cos^{-1}\left(\cos\frac{5\pi}{4}\right)$
22. $\tan^{-1}\left(\tan\frac{2\pi}{3}\right)$
23. $\sin^{-1}\left[\sin\left(-\frac{7\pi}{6}\right)\right]$
24. $\cos^{-1}\left[\cos\left(-\frac{\pi}{3}\right)\right]$
25. $\tan\left(\sin^{-1}\frac{1}{3}\right)$
26. $\tan\left(\cos^{-1}\frac{1}{3}\right)$
27. $\sec\left(\tan^{-1}\frac{1}{2}\right)$
28. $\cos\left(\sin^{-1}\frac{\sqrt{2}}{3}\right)$
29. $\cot\left[\sin^{-1}\left(-\frac{\sqrt{2}}{3}\right)\right]$
30. $\csc[\tan^{-1}(-2)]$
31. $\sin[\tan^{-1}(-3)]$
32. $\cot\left[\cos^{-1}\left(-\frac{\sqrt{3}}{3}\right)\right]$
33. $\sec\left(\sin^{-1}\frac{2\sqrt{5}}{5}\right)$
34. $\csc\left(\tan^{-1}\frac{1}{2}\right)$
35. $\sin^{-1}\left(\cos\frac{3\pi}{4}\right)$
36. $\cos^{-1}\left(\sin\frac{7\pi}{6}\right)$

In Problems 37–44, find the exact value of each expression.

37. $\cot^{-1}\sqrt{3}$
38. $\cot^{-1}1$
39. $\csc^{-1}(-1)$
40. $\csc^{-1}\sqrt{2}$
41. $\sec^{-1}\frac{2\sqrt{3}}{3}$
42. $\sec^{-1}(-2)$
43. $\cot^{-1}\left(-\frac{\sqrt{3}}{3}\right)$
44. $\csc^{-1}\left(-\frac{2\sqrt{3}}{3}\right)$

In Problems 45–56, use a calculator to find the value of each expression rounded to two decimal places.

45. $\sec^{-1}4$
46. $\csc^{-1}5$
47. $\cot^{-1}2$
48. $\sec^{-1}(-3)$

49. $\csc^{-1}(-3)$ **50.** $\cot^{-1}\left(-\frac{1}{2}\right)$ **51.** $\cot^{-1}(-\sqrt{5})$ **52.** $\cot^{-1}(-8.1)$

53. $\csc^{-1}\left(-\frac{3}{2}\right)$ **54.** $\sec^{-1}\left(-\frac{4}{3}\right)$ **55.** $\cot^{-1}\left(-\frac{3}{2}\right)$ **56.** $\cot^{-1}(-\sqrt{10})$

In Problems 57–66, write each trigonometric expression as an algebraic expression in u.

57. $\cos(\tan^{-1} u)$ **58.** $\sin(\cos^{-1} u)$ **59.** $\tan(\sin^{-1} u)$ **60.** $\tan(\cos^{-1} u)$ **61.** $\sin(\sec^{-1} u)$

62. $\sin(\cot^{-1} u)$ **63.** $\cos(\csc^{-1} u)$ **64.** $\cos(\sec^{-1} u)$ **65.** $\tan(\cot^{-1} u)$ **66.** $\tan(\sec^{-1} u)$

Applications and Extensions

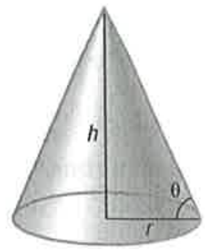
In Problems 67–78, $f(x) = \sin x$, $g(x) = \cos x$, and $h(x) = \tan x$. Find the exact value of each composite function.

67. $g\left(f^{-1}\left(\frac{12}{13}\right)\right)$ **68.** $f\left(g^{-1}\left(\frac{5}{13}\right)\right)$ **69.** $g^{-1}\left(f\left(\frac{7\pi}{4}\right)\right)$ **70.** $f^{-1}\left(g\left(\frac{5\pi}{6}\right)\right)$

71. $h\left(f^{-1}\left(-\frac{3}{5}\right)\right)$ **72.** $h\left(g^{-1}\left(-\frac{4}{5}\right)\right)$ **73.** $g\left(h^{-1}\left(\frac{12}{5}\right)\right)$ **74.** $f\left(h^{-1}\left(\frac{5}{12}\right)\right)$

75. $g^{-1}\left(f\left(-\frac{4\pi}{3}\right)\right)$ **76.** $g^{-1}\left(f\left(-\frac{5\pi}{6}\right)\right)$ **77.** $h\left(g^{-1}\left(-\frac{1}{4}\right)\right)$ **78.** $h\left(f^{-1}\left(-\frac{2}{5}\right)\right)$

Problems 79 and 80 require the following discussion: When granular materials are allowed to fall freely, they form conical (cone-shaped) piles. The naturally occurring angle of slope, measured from the horizontal, at which the loose material comes to rest is called the ***angle of repose*** *and varies for different materials. The angle of repose θ is related to the height h and base radius r of the conical pile by the equation $\theta = \cot^{-1}\frac{r}{h}$. See the illustration.*



79. Angle of Repose: Deicing Salt Due to potential transportation issues (for example, frozen waterways) deicing salt used by highway departments in the Midwest must be ordered early and stored for future use. When deicing salt is stored in a pile 14 feet high, the diameter of the base of the pile is 45 feet.
(a) Find the angle of repose for deicing salt.
(b) What is the base diameter of a pile that is 17 feet high?
(c) What is the height of a pile that has a base diameter of approximately 122 feet?
Source: Salt Institute, *The Salt Storage Handbook,* 2006

80. Angle of Repose: Bunker Sand The steepness of sand bunkers on a golf course is affected by the angle of repose of the sand (a larger angle of repose allows for steeper bunkers). A freestanding pile of loose sand from a United States Golf Association (USGA) bunker had a height of 4 feet and a base diameter of approximately 6.68 feet.
(a) Find the angle of repose for USGA bunker sand.
(b) What is the height of such a pile if the diameter of the base is 8 feet?
(c) A 6-foot high pile of loose Tour Grade 50/50 sand has a base diameter of approximately 8.44 feet. Which type of sand (USGA or Tour Grade 50/50) would be better suited for steep bunkers?
Source: 2004 *Annual Report, Purdue University Turfgrass Science Program*

81. Artillery A projectile fired into the first quadrant from the origin of a coordinate system will pass through the point (x, y) at time t according to the relationship $\cot\theta = \frac{2x}{2y + gt^2}$ where θ = the angle of elevation of the launcher and g = the acceleration due to gravity = 32.2 feet/second^2. An artilleryman is firing at an enemy bunker located 2450 feet up the side of a hill that is 6175 feet away. He fires a round, and exactly 2.27 seconds later he scores a direct hit.
(a) What angle of elevation did he use?
(b) If the angle of elevation is also given by $\sec\theta = \frac{v_0 t}{x}$, where v_0 as the muzzle velocity of the weapon, find the muzzle velocity of the artillery piece he used.
Source: *www.egwald.com/geometry/projectile3d.php*

82. Using a graphing utility, graph $y = \cot^{-1}x$.

83. Using a graphing utility, graph $y = \sec^{-1}x$.

84. Using a graphing utility, graph $y = \csc^{-1}x$.

Discussion and Writing

85. Explain in your own words how you would use your calculator to find the value of $\cot^{-1} 10$.

86. Consult three books on calculus and write down the definition in each of $y = \sec^{-1} x$ and $y = \csc^{-1} x$. Compare these with the definitions given in this book.

'Are You Prepared?' Answers

1. Domain: $\left\{x \mid x \neq \text{odd integer multiples of } \frac{\pi}{2}\right\}$; range: $\{y \leq -1 \text{ or } y \geq 1\}$ **2.** True **3.** $\frac{-2\sqrt{5}}{5}$

3.3

Solution

$$\frac{\tan v + \cot v}{\sec v \csc v} = \frac{\frac{\sin v}{\cos v} + \frac{\cos v}{\sin v}}{\frac{1}{\cos v} \cdot \frac{1}{\sin v}} = \frac{\frac{\sin^2 v + \cos^2 v}{\cos v \sin v}}{\frac{1}{\cos v \sin v}}$$

Change to sines and cosines. Add the quotients in the numerator.

$$= \frac{1}{\cos v \sin v} \cdot \frac{\cos v \sin v}{1} = 1$$

Divide the quotients; $\sin^2 v + \cos^2 v = 1$.

Now Work PROBLEM 69

Sometimes, multiplying the numerator and denominator by an appropriate f tor will result in a simplification.

EXAMPLE 8 Establishing an Identity

Establish the identity: $\dfrac{1 - \sin\theta}{\cos\theta} = \dfrac{\cos\theta}{1 + \sin\theta}$

Solution We start with the left side and multiply the numerator and the denominator 1 + sin θ. (Alternatively, we could multiply the numerator and denominator of right side by 1 − sin θ.)

$$\frac{1 - \sin\theta}{\cos\theta} = \frac{1 - \sin\theta}{\cos\theta} \cdot \frac{1 + \sin\theta}{1 + \sin\theta}$$ Multiply the numerator and denominator by $1 + \sin\theta$.

$$= \frac{1 - \sin^2\theta}{\cos\theta(1 + \sin\theta)}$$

$$= \frac{\cos^2\theta}{\cos\theta(1 + \sin\theta)}$$ $1 - \sin^2\theta = \cos^2\theta$

$$= \frac{\cos\theta}{1 + \sin\theta}$$ Cancel.

Now Work PROBLEM 53

Although a lot of practice is the only real way to learn how to establish ide ties, the following guidelines should prove helpful.

WARNING Be careful not to handle identities to be established as if they were conditional equations. You cannot establish an identity by such methods as adding the same expression to each side and obtaining a true statement. This practice is not allowed, because the original statement is precisely the one that you are trying to establish. You do not know until it has been established that it is, in fact, true. ■

Guidelines for Establishing Identities

1. It is almost always preferable to start with the side containing the more complicated expression.
2. Rewrite sums or differences of quotients as a single quotient.
3. Sometimes rewriting one side in terms of sine and cosine functions only will help.
4. Always keep your goal in mind. As you manipulate one side of the expression, you must keep in mind the form of the expression on the other side.

3.3 Asssess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. *True or False* $\sin^2\theta = 1 - \cos^2\theta$. (p. 129)

2. *True or False* $\sin(-\theta) + \cos(-\theta) = \cos\theta - \sin\theta$. (p. 1

3.3

Concepts and Vocabulary

3. Suppose that f and g are two functions with the same domain. If $f(x) = g(x)$ for every x in the domain, the equation is called a(n) _____. Otherwise, it is called a(n) _____ equation.

4. $\tan^2\theta - \sec^2\theta =$ ____.

5. $\cos(-\theta) - \cos\theta =$ ____.

6. ***True or False*** $\sin(-\theta) + \sin\theta = 0$ for any value of θ.

7. ***True or False*** In establishing an identity, it is often easiest to just multiply both sides by a well-chosen nonzero expression involving the variable.

8. ***True or False*** $\tan\theta \cdot \cos\theta = \sin\theta$ for any $\theta \neq (2k+1)\frac{\pi}{2}$.

Skill Building

In Problems 9–18, simplify each trigonometric expression by following the indicated direction.

9. Rewrite in terms of sine and cosine functions: $\tan\theta \cdot \csc\theta$.

10. Rewrite in terms of sine and cosine functions: $\cot\theta \cdot \sec\theta$.

11. Multiply $\dfrac{\cos\theta}{1-\sin\theta}$ by $\dfrac{1+\sin\theta}{1+\sin\theta}$.

12. Multiply $\dfrac{\sin\theta}{1+\cos\theta}$ by $\dfrac{1-\cos\theta}{1-\cos\theta}$.

13. Rewrite over a common denominator:
$$\frac{\sin\theta+\cos\theta}{\cos\theta} + \frac{\cos\theta-\sin\theta}{\sin\theta}$$

14. Rewrite over a common denominator:
$$\frac{1}{1-\cos v} + \frac{1}{1+\cos v}$$

15. Multiply and simplify: $\dfrac{(\sin\theta+\cos\theta)(\sin\theta+\cos\theta)-1}{\sin\theta\cos\theta}$

16. Multiply and simplify: $\dfrac{(\tan\theta+1)(\tan\theta+1)-\sec^2\theta}{\tan\theta}$

17. Factor and simplify: $\dfrac{3\sin^2\theta+4\sin\theta+1}{\sin^2\theta+2\sin\theta+1}$

18. Factor and simplify: $\dfrac{\cos^2\theta-1}{\cos^2\theta-\cos\theta}$

In Problems 19–98, establish each identity.

19. $\csc\theta \cdot \cos\theta = \cot\theta$

20. $\sec\theta \cdot \sin\theta = \tan\theta$

21. $1 + \tan^2(-\theta) = \sec^2\theta$

22. $1 + \cot^2(-\theta) = \csc^2\theta$

23. $\cos\theta(\tan\theta + \cot\theta) = \csc\theta$

24. $\sin\theta(\cot\theta + \tan\theta) = \sec\theta$

25. $\tan u \cot u - \cos^2 u = \sin^2 u$

26. $\sin u \csc u - \cos^2 u = \sin^2 u$

27. $(\sec\theta - 1)(\sec\theta + 1) = \tan^2\theta$

28. $(\csc\theta - 1)(\csc\theta + 1) = \cot^2\theta$

29. $(\sec\theta + \tan\theta)(\sec\theta - \tan\theta) = 1$

30. $(\csc\theta + \cot\theta)(\csc\theta - \cot\theta) = 1$

31. $\cos^2\theta(1 + \tan^2\theta) = 1$

32. $(1 - \cos^2\theta)(1 + \cot^2\theta) = 1$

33. $(\sin\theta + \cos\theta)^2 + (\sin\theta - \cos\theta)^2 = 2$

34. $\tan^2\theta\cos^2\theta + \cot^2\theta\sin^2\theta = 1$

35. $\sec^4\theta - \sec^2\theta = \tan^4\theta + \tan^2\theta$

36. $\csc^4\theta - \csc^2\theta = \cot^4\theta + \cot^2\theta$

37. $\sec u - \tan u = \dfrac{\cos u}{1+\sin u}$

38. $\csc u - \cot u = \dfrac{\sin u}{1+\cos u}$

39. $3\sin^2\theta + 4\cos^2\theta = 3 + \cos^2\theta$

40. $9\sec^2\theta - 5\tan^2\theta = 5 + 4\sec^2\theta$

41. $1 - \dfrac{\cos^2\theta}{1+\sin\theta} = \sin\theta$

42. $1 - \dfrac{\sin^2\theta}{1-\cos\theta} = -\cos\theta$

43. $\dfrac{1+\tan v}{1-\tan v} = \dfrac{\cot v+1}{\cot v-1}$

44. $\dfrac{\csc v-1}{\csc v+1} = \dfrac{1-\sin v}{1+\sin v}$

45. $\dfrac{\sec\theta}{\csc\theta} + \dfrac{\sin\theta}{\cos\theta} = 2\tan\theta$

46. $\dfrac{\csc\theta-1}{\cot\theta} = \dfrac{\cot\theta}{\csc\theta+1}$

47. $\dfrac{1+\sin\theta}{1-\sin\theta} = \dfrac{\csc\theta+1}{\csc\theta-1}$

48. $\dfrac{\cos\theta+1}{\cos\theta-1} = \dfrac{1+\sec\theta}{1-\sec\theta}$

49. $\dfrac{1-\sin v}{\cos v} + \dfrac{\cos v}{1-\sin v} = 2\sec v$

50. $\dfrac{\cos v}{1+\sin v} + \dfrac{1+\sin v}{\cos v} = 2\sec v$

51. $\dfrac{\sin\theta}{\sin\theta-\cos\theta} = \dfrac{1}{1-\cot\theta}$

52. $1 - \dfrac{\sin^2\theta}{1+\cos\theta} = \cos\theta$

53. $\dfrac{1-\sin\theta}{1+\sin\theta} = (\sec\theta - \tan\theta)^2$

54. $\dfrac{1-\cos\theta}{1+\cos\theta} = (\csc\theta - \cot\theta)^2$

55. $\dfrac{\cos\theta}{1-\tan\theta} + \dfrac{\sin\theta}{1-\cot\theta} = \sin\theta + \cos\theta$

56. $\dfrac{\cot\theta}{1-\tan\theta} + \dfrac{\tan\theta}{1-\cot\theta} = 1 + \tan\theta + \cot\theta$

3.3

57. $\tan\theta + \dfrac{\cos\theta}{1+\sin\theta} = \sec\theta$

58. $\dfrac{\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta} = \dfrac{\tan\theta}{1-\tan^2\theta}$

59. $\dfrac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1} = \tan\theta + \sec\theta$

60. $\dfrac{\sin\theta - \cos\theta + 1}{\sin\theta + \cos\theta - 1} = \dfrac{\sin\theta + 1}{\cos\theta}$

61. $\dfrac{\tan\theta - \cot\theta}{\tan\theta + \cot\theta} = \sin^2\theta - \cos^2\theta$

62. $\dfrac{\sec\theta - \cos\theta}{\sec\theta + \cos\theta} = \dfrac{\sin^2\theta}{1+\cos^2\theta}$

63. $\dfrac{\tan u - \cot u}{\tan u + \cot u} + 1 = 2\sin^2 u$

64. $\dfrac{\tan u - \cot u}{\tan u + \cot u} + 2\cos^2 u = 1$

65. $\dfrac{\sec\theta + \tan\theta}{\cot\theta + \cos\theta} = \tan\theta\sec\theta$

66. $\dfrac{\sec\theta}{1+\sec\theta} = \dfrac{1-\cos\theta}{\sin^2\theta}$

67. $\dfrac{1-\tan^2\theta}{1+\tan^2\theta} + 1 = 2\cos^2\theta$

68. $\dfrac{1-\cot^2\theta}{1+\cot^2\theta} + 2\cos^2\theta = 1$

69. $\dfrac{\sec\theta - \csc\theta}{\sec\theta\csc\theta} = \sin\theta - \cos\theta$

70. $\dfrac{\sin^2\theta - \tan\theta}{\cos^2\theta - \cot\theta} = \tan^2\theta$

71. $\sec\theta - \cos\theta = \sin\theta\tan\theta$

72. $\tan\theta + \cot\theta = \sec\theta\csc\theta$

73. $\dfrac{1}{1-\sin\theta} + \dfrac{1}{1+\sin\theta} = 2\sec^2\theta$

74. $\dfrac{1+\sin\theta}{1-\sin\theta} - \dfrac{1-\sin\theta}{1+\sin\theta} = 4\tan\theta\,$se

75. $\dfrac{\sec\theta}{1-\sin\theta} = \dfrac{1+\sin\theta}{\cos^3\theta}$

76. $\dfrac{1+\sin\theta}{1-\sin\theta} = (\sec\theta + \tan\theta)^2$

77. $\dfrac{(\sec v - \tan v)^2 + 1}{\csc v(\sec v - \tan v)} = 2\tan v$

78. $\dfrac{\sec^2 v - \tan^2 v + \tan v}{\sec v} = \sin v + \cos v$

79. $\dfrac{\sin\theta + \cos\theta}{\cos\theta} - \dfrac{\sin\theta - \cos\theta}{\sin\theta} = \sec\theta\csc\theta$

80. $\dfrac{\sin\theta + \cos\theta}{\sin\theta} - \dfrac{\cos\theta - \sin\theta}{\cos\theta} = \sec\theta\csc\theta$

81. $\dfrac{\sin^3\theta + \cos^3\theta}{\sin\theta + \cos\theta} = 1 - \sin\theta\cos\theta$

82. $\dfrac{\sin^3\theta + \cos^3\theta}{1 - 2\cos^2\theta} = \dfrac{\sec\theta - \sin\theta}{\tan\theta - 1}$

83. $\dfrac{\cos^2\theta - \sin^2\theta}{1-\tan^2\theta} = \cos^2\theta$

84. $\dfrac{\cos\theta + \sin\theta - \sin^3\theta}{\sin\theta} = \cot\theta + \cos^2\theta$

85. $\dfrac{(2\cos^2\theta - 1)^2}{\cos^4\theta - \sin^4\theta} = 1 - 2\sin^2\theta$

86. $\dfrac{1 - 2\cos^2\theta}{\sin\theta\cos\theta} = \tan\theta - \cot\theta$

87. $\dfrac{1+\sin\theta+\cos\theta}{1+\sin\theta-\cos\theta} = \dfrac{1+\cos\theta}{\sin\theta}$

88. $\dfrac{1+\cos\theta+\sin\theta}{1+\cos\theta-\sin\theta} = \sec\theta + \tan\theta$

89. $(a\sin\theta + b\cos\theta)^2 + (a\cos\theta - b\sin\theta)^2 = a^2 + b^2$

90. $(2a\sin\theta\cos\theta)^2 + a^2(\cos^2\theta - \sin^2\theta)^2 = a^2$

91. $\dfrac{\tan\alpha + \tan\beta}{\cot\alpha + \cot\beta} = \tan\alpha\tan\beta$

92. $(\tan\alpha + \tan\beta)(1 - \cot\alpha\cot\beta) + (\cot\alpha + \cot\beta)(1 - \tan\alpha\tan\beta) = 0$

93. $(\sin\alpha + \cos\beta)^2 + (\cos\beta + \sin\alpha)(\cos\beta - \sin\alpha) = 2\cos\beta(\sin\alpha + \cos\beta)$

94. $(\sin\alpha - \cos\beta)^2 + (\cos\beta + \sin\alpha)(\cos\beta - \sin\alpha) = -2\cos\beta(\sin\alpha - \cos\beta)$

95. $\ln|\sec\theta| = -\ln|\cos\theta|$

96. $\ln|\tan\theta| = \ln|\sin\theta| - \ln|\cos\theta|$

97. $\ln|1+\cos\theta| + \ln|1-\cos\theta| = 2\ln|\sin\theta|$

98. $\ln|\sec\theta + \tan\theta| + \ln|\sec\theta - \tan\theta| = 0$

In Problems 99–102, show that the functions f and g are identically equal.

99. $f(x) = \sin x \cdot \tan x \quad g(x) = \sec x - \cos x$

100. $f(x) = \cos x \cdot \cot x \quad g(x) = \csc x - \sin x$

101. $f(\theta) = \dfrac{1-\sin\theta}{\cos\theta} - \dfrac{\cos\theta}{1+\sin\theta} \quad g(\theta) = 0$

102. $f(\theta) = \tan\theta + \sec\theta \quad g(\theta) = \dfrac{\cos\theta}{1-\sin\theta}$

Since $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$, we know that $\cos\alpha \ge 0$. As a result,

$$\cos\alpha = \sqrt{1 - \sin^2\alpha} = \sqrt{1 - u^2}$$

Similarly, since $0 \le \beta \le \pi$, we know that $\sin\beta \ge 0$. Then

$$\sin\beta = \sqrt{1 - \cos^2\beta} = \sqrt{1 - v^2}$$

As a result,

$$\sin(\sin^{-1} u + \cos^{-1} v) = \sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$= uv + \sqrt{1 - u^2}\sqrt{1 - v^2}$$

Now Work PROBLEM 83

SUMMARY Sum and Difference Formulas

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \qquad \cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$
$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \qquad \sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \qquad \tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

3.4 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. The distance d from the point $(2, -3)$ to the point $(5, 1)$ is _____. (p. 3)
2. If $\sin\theta = \frac{4}{5}$ and θ is in quadrant II, then $\cos\theta =$ _____. (pp. 130–132)
3. (a) $\sin\frac{\pi}{4}\cdot\cos\frac{\pi}{3} =$ _____. (pp. 112, 115)

   (b) $\tan\frac{\pi}{4} - \sin\frac{\pi}{6} =$ _____. (pp. 112, 115)

Concepts and Vocabulary

4. $\cos(\alpha + \beta) = \cos\alpha\cos\beta$ ____ $\sin\alpha\sin\beta$.
5. $\sin(\alpha - \beta) = \sin\alpha\cos\beta$ ____ $\cos\alpha\sin\beta$.
6. ***True or False*** $\sin(\alpha + \beta) = \sin\alpha + \sin\beta + 2\sin\alpha\sin\beta$
7. ***True or False*** $\tan 75° = \tan 30° + \tan 45°$
8. ***True or False*** $\cos\left(\frac{\pi}{2} - \theta\right) = \cos\theta$

Skill Building

In Problems 9–20, find the exact value of each expression.

9. $\sin\frac{5\pi}{12}$
10. $\sin\frac{\pi}{12}$
11. $\cos\frac{7\pi}{12}$
12. $\tan\frac{7\pi}{12}$
13. $\cos 165°$
14. $\sin 105°$
15. $\tan 15°$
16. $\tan 195°$
17. $\sin\frac{17\pi}{12}$
18. $\tan\frac{19\pi}{12}$
19. $\sec\left(-\frac{\pi}{12}\right)$
20. $\cot\left(-\frac{5\pi}{12}\right)$

In Problems 21–30, find the exact value of each expression.

21. $\sin 20°\cos 10° + \cos 20°\sin 10°$
22. $\sin 20°\cos 80° - \cos 20°\sin 80°$
23. $\cos 70°\cos 20° - \sin 70°\sin 20°$
24. $\cos 40°\cos 10° + \sin 40°\sin 10°$
25. $\dfrac{\tan 20° + \tan 25°}{1 - \tan 20°\tan 25°}$
26. $\dfrac{\tan 40° - \tan 10°}{1 + \tan 40°\tan 10°}$
27. $\sin\frac{\pi}{12}\cos\frac{7\pi}{12} - \cos\frac{\pi}{12}\sin\frac{7\pi}{12}$
28. $\cos\frac{5\pi}{12}\cos\frac{7\pi}{12} - \sin\frac{5\pi}{12}\sin\frac{7\pi}{12}$
29. $\cos\frac{\pi}{12}\cos\frac{5\pi}{12} + \sin\frac{5\pi}{12}\sin\frac{\pi}{12}$
30. $\sin\frac{\pi}{18}\cos\frac{5\pi}{18} + \cos\frac{\pi}{18}\sin\frac{5\pi}{18}$

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In Problems 31–36, find the exact value of each of the following under the given conditions:

(a) $\sin(\alpha + \beta)$ (b) $\cos(\alpha + \beta)$ (c) $\sin(\alpha - \beta)$ (d) $\tan(\alpha - \beta)$

31. $\sin\alpha = \frac{3}{5}, 0 < \alpha < \frac{\pi}{2}$; $\cos\beta = \frac{2\sqrt{5}}{5}, -\frac{\pi}{2} < \beta < 0$

32. $\cos\alpha = \frac{\sqrt{5}}{5}, 0 < \alpha < \frac{\pi}{2}$; $\sin\beta = -\frac{4}{5}, -\frac{\pi}{2} < \beta < 0$

33. $\tan\alpha = -\frac{4}{3}, \frac{\pi}{2} < \alpha < \pi$; $\cos\beta = \frac{1}{2}, 0 < \beta < \frac{\pi}{2}$

34. $\tan\alpha = \frac{5}{12}, \pi < \alpha < \frac{3\pi}{2}$; $\sin\beta = -\frac{1}{2}, \pi < \beta < \frac{3\pi}{2}$

35. $\sin\alpha = \frac{5}{13}, -\frac{3\pi}{2} < \alpha < -\pi$; $\tan\beta = -\sqrt{3}, \frac{\pi}{2} < \beta < \pi$

36. $\cos\alpha = \frac{1}{2}, -\frac{\pi}{2} < \alpha < 0$; $\sin\beta = \frac{1}{3}, 0 < \beta < \frac{\pi}{2}$

37. If $\sin\theta = \frac{1}{3}$, θ in quadrant II, find the exact value of:

(a) $\cos\theta$ (b) $\sin\left(\theta + \frac{\pi}{6}\right)$ (c) $\cos\left(\theta - \frac{\pi}{3}\right)$ (d) $\tan\left(\theta + \frac{\pi}{4}\right)$

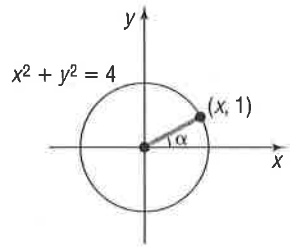
38. If $\cos\theta = \frac{1}{4}$, θ in quadrant IV, find the exact value of:

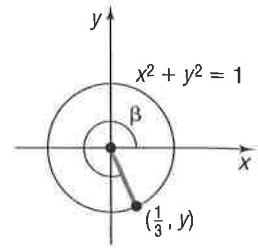
(a) $\sin\theta$ (b) $\sin\left(\theta - \frac{\pi}{6}\right)$ (c) $\cos\left(\theta + \frac{\pi}{3}\right)$ (d) $\tan\left(\theta - \frac{\pi}{4}\right)$

In Problems 39–44, use the figures to evaluate each function if $f(x) = \sin x$, $g(x) = \cos x$, and $h(x) = \tan x$.

39. $f(\alpha + \beta)$ **40.** $g(\alpha + \beta)$ **41.** $g(\alpha - \beta)$

42. $f(\alpha - \beta)$ **43.** $h(\alpha + \beta)$ **44.** $h(\alpha - \beta)$





In Problems 45–70, establish each identity.

45. $\sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta$

46. $\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$

47. $\sin(\pi - \theta) = \sin\theta$

48. $\cos(\pi - \theta) = -\cos\theta$

49. $\sin(\pi + \theta) = -\sin\theta$

50. $\cos(\pi + \theta) = -\cos\theta$

51. $\tan(\pi - \theta) = -\tan\theta$

52. $\tan(2\pi - \theta) = -\tan\theta$

53. $\sin\left(\frac{3\pi}{2} + \theta\right) = -\cos\theta$

54. $\cos\left(\frac{3\pi}{2} + \theta\right) = \sin\theta$

55. $\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2\sin\alpha\cos\beta$

56. $\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2\cos\alpha\cos\beta$

57. $\frac{\sin(\alpha + \beta)}{\sin\alpha\cos\beta} = 1 + \cot\alpha\tan\beta$

58. $\frac{\sin(\alpha + \beta)}{\cos\alpha\cos\beta} = \tan\alpha + \tan\beta$

59. $\frac{\cos(\alpha + \beta)}{\cos\alpha\cos\beta} = 1 - \tan\alpha\tan\beta$

60. $\frac{\cos(\alpha - \beta)}{\sin\alpha\cos\beta} = \cot\alpha + \tan\beta$

61. $\frac{\sin(\alpha + \beta)}{\sin(\alpha - \beta)} = \frac{\tan\alpha + \tan\beta}{\tan\alpha - \tan\beta}$

62. $\frac{\cos(\alpha + \beta)}{\cos(\alpha - \beta)} = \frac{1 - \tan\alpha\tan\beta}{1 + \tan\alpha\tan\beta}$

63. $\cot(\alpha + \beta) = \frac{\cot\alpha\cot\beta - 1}{\cot\beta + \cot\alpha}$

64. $\cot(\alpha - \beta) = \frac{\cot\alpha\cot\beta + 1}{\cot\beta - \cot\alpha}$

65. $\sec(\alpha + \beta) = \frac{\csc\alpha\csc\beta}{\cot\alpha\cot\beta - 1}$

66. $\sec(\alpha - \beta) = \frac{\sec\alpha\sec\beta}{1 + \tan\alpha\tan\beta}$

67. $\sin(\alpha - \beta)\sin(\alpha + \beta) = \sin^2\alpha - \sin^2\beta$

68. $\cos(\alpha - \beta)\cos(\alpha + \beta) = \cos^2\alpha - \sin^2\beta$

69. $\sin(\theta + k\pi) = (-1)^k\sin\theta$, k any integer

70. $\cos(\theta + k\pi) = (-1)^k\cos\theta$, k any integer

In Problems 71–82, find the exact value of each expression.

71. $\sin\left(\sin^{-1}\frac{1}{2} + \cos^{-1}0\right)$

72. $\sin\left(\sin^{-1}\frac{\sqrt{3}}{2} + \cos^{-1}1\right)$

73. $\sin\left[\sin^{-1}\frac{3}{5} - \cos^{-1}\left(-\frac{4}{5}\right)\right]$

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74. $\sin\left[\sin^{-1}\left(-\frac{4}{5}\right) - \tan^{-1}\frac{3}{4}\right]$

75. $\cos\left(\tan^{-1}\frac{4}{3} + \cos^{-1}\frac{5}{13}\right)$

76. $\cos\left[\tan^{-1}\frac{5}{12} - \sin^{-1}\left(-\frac{3}{5}\right)\right]$

77. $\cos\left(\sin^{-1}\frac{5}{13} - \tan^{-1}\frac{3}{4}\right)$

78. $\cos\left(\tan^{-1}\frac{4}{3} + \cos^{-1}\frac{12}{13}\right)$

79. $\tan\left(\sin^{-1}\frac{3}{5} + \frac{\pi}{6}\right)$

80. $\tan\left(\frac{\pi}{4} - \cos^{-1}\frac{3}{5}\right)$

81. $\tan\left(\sin^{-1}\frac{4}{5} + \cos^{-1}1\right)$

82. $\tan\left(\cos^{-1}\frac{4}{5} + \sin^{-1}1\right)$

In Problems 83–88, write each trigonometric expression as an algebraic expression containing u and v. Give the restrictions required u and v.

83. $\cos(\cos^{-1} u + \sin^{-1} v)$

84. $\sin(\sin^{-1} u - \cos^{-1} v)$

85. $\sin(\tan^{-1} u - \sin^{-1} v)$

86. $\cos(\tan^{-1} u + \tan^{-1} v)$

87. $\tan(\sin^{-1} u - \cos^{-1} v)$

88. $\sec(\tan^{-1} u + \cos^{-1} v)$

Applications and Extensions

89. Show that $\sin^{-1} v + \cos^{-1} v = \frac{\pi}{2}$.

90. Show that $\tan^{-1} v + \cot^{-1} v = \frac{\pi}{2}$.

91. Show that $\tan^{-1}\left(\frac{1}{v}\right) = \frac{\pi}{2} - \tan^{-1} v$, if $v > 0$.

92. Show that $\cot^{-1} e^{v} = \tan^{-1} e^{-v}$.

93. Show that $\sin(\sin^{-1} v + \cos^{-1} v) = 1$.

94. Show that $\cos(\sin^{-1} v + \cos^{-1} v) = 0$.

95. Calculus Show that the difference quotient for $f(x) = \sin x$ is given by

$$\frac{f(x+h) - f(x)}{h} = \frac{\sin(x+h) - \sin x}{h} = \cos x \cdot \frac{\sin h}{h} - \sin x \cdot \frac{1 - \cos h}{h}$$

96. Calculus Show that the difference quotient for $f(x) = \cos x$ is given by

$$\frac{f(x+h) - f(x)}{h} = \frac{\cos(x+h) - \cos x}{h} = -\sin x \cdot \frac{\sin h}{h} - \cos x \cdot \frac{1 - \cos h}{h}$$

97. One, Two, Three

(a) Show that $\tan(\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3) = 0$.

(b) Conclude from part (a) that

$$\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 = \pi.$$

Source: *College Mathematics Journal,* Vol. 37, No. 3, May 2006

98. Electric Power In an alternating current (ac) circuit, the instantaneous power p at time t is given by

$$p(t) = V_m I_m \cos\phi \sin^2(\omega t) - V_m I_m \sin\phi \sin(\omega t)\cos(\omega t)$$

Show that this is equivalent to

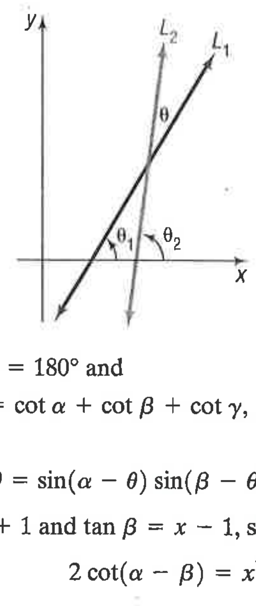
$$p(t) = V_m I_m \sin(\omega t)\sin(\omega t - \phi)$$

Source: *HyperPhysics, hosted by Georgia State University*

99. Geometry: Angle Between Two Lines Let L_1 and L_2 c note two nonvertical intersecting lines, and let θ denote t acute angle between L_1 and L_2 (see the figure). Show tha

$$\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2}$$

where m_1 and m_2 are the slopes of L_1 and L_2, respectivel [**Hint:** Use the facts that $\tan\theta_1 = m_1$ and $\tan\theta_2 = m_2$.]



100. If $\alpha + \beta + \gamma = 180°$ and

$$\cot\theta = \cot\alpha + \cot\beta + \cot\gamma, \quad 0 < \theta < 90°$$

show that

$$\sin^3\theta = \sin(\alpha - \theta)\sin(\beta - \theta)\sin(\gamma - \theta)$$

101. If $\tan\alpha = x + 1$ and $\tan\beta = x - 1$, show that

$$2\cot(\alpha - \beta) = x^2$$

Discussion and Writing

102. Discuss the following derivation:

$$\tan\left(\theta + \frac{\pi}{2}\right) = \frac{\tan\theta + \tan\frac{\pi}{2}}{1 - \tan\theta\tan\frac{\pi}{2}} = \frac{\frac{\tan\theta}{\tan\frac{\pi}{2}} + 1}{\frac{1}{\tan\frac{\pi}{2}} - \tan\theta} = \frac{0 + 1}{0 - \tan\theta} = \frac{1}{-\tan\theta} = -\cot\theta$$

Can you justify each step?

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and

$$\sin\alpha = \sin\left[2\left(\frac{\alpha}{2}\right)\right] = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} \quad \text{Double-angle Formula}$$

Then

$$\frac{1-\cos\alpha}{\sin\alpha} = \frac{2\sin^2\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}} = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} = \tan\frac{\alpha}{2}$$

Since it also can be shown that

$$\frac{1-\cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1+\cos\alpha}$$

we have the following two Half-angle Formulas:

Half-angle Formulas for $\tan\frac{\alpha}{2}$

$$\tan\frac{\alpha}{2} = \frac{1-\cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1+\cos\alpha} \qquad \textbf{(11)}$$

With this formula, the solution to Example 6(c) can be obtained as follows:

$$\cos\alpha = -\frac{3}{5} \quad \pi < \alpha < \frac{3\pi}{2}$$

$$\sin\alpha = -\sqrt{1-\cos^2\alpha} = -\sqrt{1-\frac{9}{25}} = -\sqrt{\frac{16}{25}} = -\frac{4}{5}$$

Then, by equation (11),

$$\tan\frac{\alpha}{2} = \frac{1-\cos\alpha}{\sin\alpha} = \frac{1-\left(-\frac{3}{5}\right)}{-\frac{4}{5}} = \frac{\frac{8}{5}}{-\frac{4}{5}} = -2$$

3.5 Assess Your Understanding

Concepts and Vocabulary

1. $\cos(2\theta) = \cos^2\theta - $ ____ $=$ ____ $-1 = 1 -$ ____.
2. $\sin^2\frac{\theta}{2} = \frac{\_\_\_}{2}$.
3. $\tan\frac{\theta}{2} = \frac{1-\cos\theta}{\_\_\_}$.
4. ***True or False*** $\cos(2\theta)$ has three equivalent forms: $\cos^2\theta - \sin^2\theta,\ 1-2\sin^2\theta,\ 2\cos^2\theta - 1$
5. ***True or False*** $\sin(2\theta)$ has two equivalent forms: $2\sin\theta\cos\theta$ and $\sin^2\theta - \cos^2\theta$
6. ***True or False*** $\tan(2\theta) + \tan(2\theta) = \tan(4\theta)$

Skill Building

In Problems 7–18, use the information given about the angle θ, $0 \le \theta < 2\pi$, to find the exact value of

(a) $\sin(2\theta)$ *(b)* $\cos(2\theta)$ *(c)* $\sin\frac{\theta}{2}$ *(d)* $\cos\frac{\theta}{2}$

7. $\sin\theta = \frac{3}{5} \quad 0 < \theta < \frac{\pi}{2}$
8. $\cos\theta = \frac{3}{5} \quad 0 < \theta < \frac{\pi}{2}$
9. $\tan\theta = \frac{4}{3} \quad \pi < \theta < \frac{3\pi}{2}$
10. $\tan\theta = \frac{1}{2} \quad \pi < \theta < \frac{3\pi}{2}$
11. $\cos\theta = -\frac{\sqrt{6}}{3} \quad \frac{\pi}{2} < \theta < \pi$
12. $\sin\theta = -\frac{\sqrt{3}}{3} \quad \frac{3\pi}{2} < \theta < 2\pi$

3.5

13. $\sec\theta = 3 \quad \sin\theta > 0$

14. $\csc\theta = -\sqrt{5} \quad \cos\theta < 0$

15. $\cot\theta = -2 \quad \sec\theta < 0$

16. $\sec\theta = 2 \quad \csc\theta < 0$

17. $\tan\theta = -3 \quad \sin\theta < 0$

18. $\cot\theta = 3 \quad \cos\theta < 0$

In Problems 19–28, use the Half-angle Formulas to find the exact value of each expression.

19. $\sin 22.5°$

20. $\cos 22.5°$

21. $\tan \frac{7\pi}{8}$

22. $\tan \frac{9\pi}{8}$

23. $\cos 165°$

24. $\sin 195°$

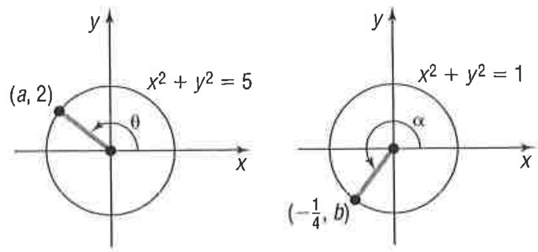
25. $\sec \frac{15\pi}{8}$

26. $\csc \frac{7\pi}{8}$

27. $\sin\left(-\frac{\pi}{8}\right)$

28. $\cos\left(-\frac{3\pi}{8}\right)$

In Problems 29–40, use the figures to evaluate each function given that $f(x) = \sin x$, $g(x) = \cos x$, and $h(x) = \tan x$.



29. $f(2\theta)$

30. $g(2\theta)$

31. $g\left(\frac{\theta}{2}\right)$

32. $f\left(\frac{\theta}{2}\right)$

33. $h(2\theta)$

34. $h\left(\frac{\theta}{2}\right)$

35. $g(2\alpha)$

36. $f(2\alpha)$

37. $f\left(\frac{\alpha}{2}\right)$

38. $g\left(\frac{\alpha}{2}\right)$

39. $h\left(\frac{\alpha}{2}\right)$

40. $h(2\alpha)$

41. Show that $\sin^4\theta = \frac{3}{8} - \frac{1}{2}\cos(2\theta) + \frac{1}{8}\cos(4\theta)$.

42. Show that $\sin(4\theta) = (\cos\theta)(4\sin\theta - 8\sin^3\theta)$.

43. Develop a formula for $\cos(3\theta)$ as a third-degree polynomial in the variable $\cos\theta$.

44. Develop a formula for $\cos(4\theta)$ as a fourth-degree polynomial in the variable $\cos\theta$.

45. Find an expression for $\sin(5\theta)$ as a fifth-degree polynomial in the variable $\sin\theta$.

46. Find an expression for $\cos(5\theta)$ as a fifth-degree polynomial in the variable $\cos\theta$.

In Problems 47–68, establish each identity.

47. $\cos^4\theta - \sin^4\theta = \cos(2\theta)$

48. $\dfrac{\cot\theta - \tan\theta}{\cot\theta + \tan\theta} = \cos(2\theta)$

49. $\cot(2\theta) = \dfrac{\cot^2\theta - 1}{2\cot\theta}$

50. $\cot(2\theta) = \frac{1}{2}(\cot\theta - \tan\theta)$

51. $\sec(2\theta) = \dfrac{\sec^2\theta}{2 - \sec^2\theta}$

52. $\csc(2\theta) = \frac{1}{2}\sec\theta\csc\theta$

53. $\cos^2(2u) - \sin^2(2u) = \cos(4u)$

54. $(4\sin u\cos u)(1 - 2\sin^2 u) = \sin(4u)$

55. $\dfrac{\cos(2\theta)}{1 + \sin(2\theta)} = \dfrac{\cot\theta - 1}{\cot\theta + 1}$

56. $\sin^2\theta\cos^2\theta = \frac{1}{8}[1 - \cos(4\theta)]$

57. $\sec^2\frac{\theta}{2} = \dfrac{2}{1 + \cos\theta}$

58. $\csc^2\frac{\theta}{2} = \dfrac{2}{1 - \cos\theta}$

59. $\cot^2\frac{v}{2} = \dfrac{\sec v + 1}{\sec v - 1}$

60. $\tan\frac{v}{2} = \csc v - \cot v$

61. $\cos\theta = \dfrac{1 - \tan^2\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}}$

62. $1 - \frac{1}{2}\sin(2\theta) = \dfrac{\sin^3\theta + \cos^3\theta}{\sin\theta + \cos\theta}$

63. $\dfrac{\sin(3\theta)}{\sin\theta} - \dfrac{\cos(3\theta)}{\cos\theta} = 2$

64. $\dfrac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta} - \dfrac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta} = 2\tan(2\theta)$

65. $\tan(3\theta) = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

66. $\tan\theta + \tan(\theta + 120°) + \tan(\theta + 240°) = 3\tan(3\theta)$

67. $\ln|\sin\theta| = \frac{1}{2}(\ln|1 - \cos(2\theta)| - \ln 2)$

68. $\ln|\cos\theta| = \frac{1}{2}(\ln|1 + \cos(2\theta)| - \ln 2)$

In Problems 69–80, find the exact value of each expression.

69. $\sin\left(2\sin^{-1}\frac{1}{2}\right)$ **70.** $\sin\left[2\sin^{-1}\frac{\sqrt{3}}{2}\right]$ **71.** $\cos\left(2\sin^{-1}\frac{3}{5}\right)$ **72.** $\cos\left(2\cos^{-1}\frac{4}{5}\right)$

73. $\tan\left[2\cos^{-1}\left(-\frac{3}{5}\right)\right]$ **74.** $\tan\left(2\tan^{-1}\frac{3}{4}\right)$ **75.** $\sin\left(2\cos^{-1}\frac{4}{5}\right)$ **76.** $\cos\left[2\tan^{-1}\left(-\frac{4}{3}\right)\right]$

77. $\sin^2\left(\frac{1}{2}\cos^{-1}\frac{3}{5}\right)$ **78.** $\cos^2\left(\frac{1}{2}\sin^{-1}\frac{3}{5}\right)$ **79.** $\sec\left(2\tan^{-1}\frac{3}{4}\right)$ **80.** $\csc\left[2\sin^{-1}\left(-\frac{3}{5}\right)\right]$

Applications and Extensions

81. Laser Projection In a laser projection system, the **optical** or **scanning angle** θ is related to the throw distance D from the scanner to the screen and the projected image width W by the equation $D = \dfrac{\frac{1}{2}W}{\csc\theta - \cot\theta}$.

(a) Show that the projected image width is given by $W = 2D\tan\frac{\theta}{2}$.

(b) Find the optical angle if the throw distance is 15 feet and the projected image width is 6.5 feet.

Source: *Pangolin Laser Systems, Inc.*

82. Product of Inertia The **product of inertia** for an area about inclined axes is given by the formula

$$I_{uv} = I_x \sin\theta\cos\theta - I_y\sin\theta\cos\theta + I_{xy}(\cos^2\theta - \sin^2\theta).$$

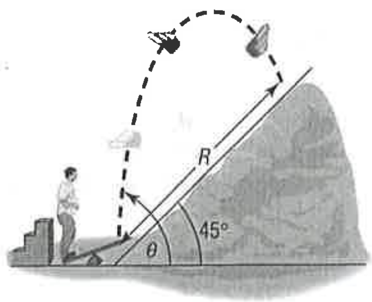
Show that this is equivalent to

$$I_{uv} = \frac{I_x - I_y}{2}\sin 2\theta + I_{xy}\cos 2\theta.$$

Source: Adapted from Hibbeler, *Engineering Mechanics: Statics*, 10th ed., Prentice Hall © 2004.

83. Projectile Motion An object is propelled upward at an angle θ, $45° < \theta < 90°$, to the horizontal with an initial velocity of v_0 feet per second from the base of a plane that makes an angle of 45°; with the horizontal. See the illustration. If air resistance is ignored, the distance R that it travels up the inclined plane is given by the function

$$R(\theta) = \frac{v_0^2\sqrt{2}}{16}\cos\theta(\sin\theta - \cos\theta)$$



(a) Show that

$$R(\theta) = \frac{v_0^2\sqrt{2}}{32}[\sin(2\theta) - \cos(2\theta) - 1]$$

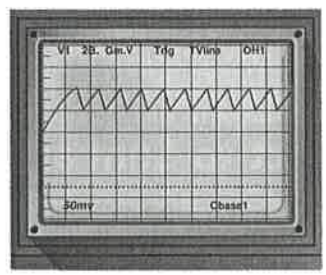
(b) Graph $R = R(\theta)$. (Use $v_0 = 32$ feet per second.)

(c) What value of θ makes R the largest? (Use $v_0 = 32$ feet per second.)

84. Sawtooth Curve An oscilloscope often displays a sawtooth curve. This curve can be approximated by sinusoidal curves of varying periods and amplitudes. A first approximation to the sawtooth curve is given by

$$y = \frac{1}{2}\sin(2\pi x) + \frac{1}{4}\sin(4\pi x)$$

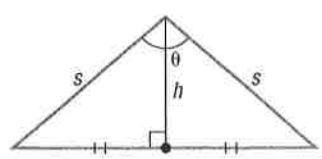
Show that $y = \sin(2\pi x)\cos^2(\pi x)$.



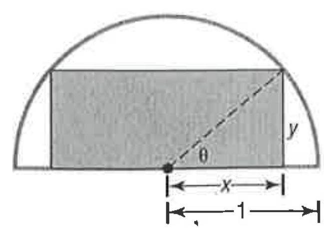
85. Area of an Isosceles Triangle Show that the area A of an isosceles triangle whose equal sides are of length s and θ is the angle between them is

$$\frac{1}{2}s^2\sin\theta$$

[**Hint:** See the illustration. The height h bisects the angle θ and is the perpendicular bisector of the base.]



86. Geometry A rectangle is inscribed in a semicircle of radius 1. See the illustration.



(a) Express the area A of the rectangle as a function of the angle θ shown in the illustration.

(b) Show that $A(\theta) = \sin(2\theta)$.

(c) Find the angle θ that results in the largest area A.

(d) Find the dimensions of this largest rectangle.

87. If $x = 2\tan\theta$, express $\sin(2\theta)$ as a function of x.

88. If $x = 2\tan\theta$, express $\cos(2\theta)$ as a function of x.

3.5

89. Find the value of the number C:

$$\frac{1}{2}\sin^2 x + C = -\frac{1}{4}\cos(2x)$$

90. Find the value of the number C:

$$\frac{1}{2}\cos^2 x + C = \frac{1}{4}\cos(2x)$$

91. If $z = \tan\frac{\alpha}{2}$, show that $\sin\alpha = \frac{2z}{1+z^2}$.

92. If $z = \tan\frac{\alpha}{2}$, show that $\cos\alpha = \frac{1-z^2}{1+z^2}$.

93. Graph $f(x) = \sin^2 x = \frac{1-\cos(2x)}{2}$ for $0 \le x \le 2\pi$ by using transformations.

94. Repeat Problem 93 for $g(x) = \cos^2 x$.

95. Use the fact that

$$\cos\frac{\pi}{12} = \frac{1}{4}\left(\sqrt{6} + \sqrt{2}\right)$$

to find $\sin\frac{\pi}{24}$ and $\cos\frac{\pi}{24}$.

96. Show that

$$\cos\frac{\pi}{8} = \frac{\sqrt{2+\sqrt{2}}}{2}$$

and use it to find $\sin\frac{\pi}{16}$ and $\cos\frac{\pi}{16}$.

97. Show that

$$\sin^3\theta + \sin^3(\theta + 120°) + \sin^3(\theta + 240°) = -\frac{3}{4}\sin(3\theta)$$

98. If $\tan\theta = a\tan\frac{\theta}{3}$, express $\tan\frac{\theta}{3}$ in terms of a.

Discussion and Writing

99. Go to the library and research Chebyshëv polynomials. Write a report on your findings.

3.6 Product-to-Sum and Sum-to-Product Formulas

OBJECTIVES 1 Express Products as Sums (p. 222)
2 Express Sums as Products (p. 223)

1 Express Products as Sums

Sum and difference formulas can be used to derive formulas for writing the products of sines and/or cosines as sums or differences. These identities are usually called the **Product-to-Sum Formulas.**

THEOREM **Product-to-Sum Formulas**

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)] \quad (1)$$

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)] \quad (2)$$

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)] \quad (3)$$

These formulas do not have to be memorized. Instead, you should remember how they are derived. Then, when you want to use them, either look them up or derive them, as needed.

To derive formulas (1) and (2), write down the sum and difference formulas for the cosine:

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \quad (4)$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \quad (5)$$

Subtract equation (5) from equation (4) to get

$$\cos(\alpha-\beta) - \cos(\alpha+\beta) = 2\sin\alpha\sin\beta$$

from which

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$$

3.6

Proof

$$2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} = 2\cdot\frac{1}{2}\left[\sin\left(\frac{\alpha+\beta}{2}+\frac{\alpha-\beta}{2}\right)+\sin\left(\frac{\alpha+\beta}{2}-\frac{\alpha-\beta}{2}\right)\right]$$

↑ Product-to-Sum Formula (3)

$$= \sin\frac{2\alpha}{2} + \sin\frac{2\beta}{2} = \sin\alpha + \sin\beta$$

EXAMPLE 2 **Expressing Sums (or Differences) as a Product**

Express each sum or difference as a product of sines and/or cosines.

(a) $\sin(5\theta) - \sin(3\theta)$ (b) $\cos(3\theta) + \cos(2\theta)$

Solution (a) We use formula (7) to get

$$\sin(5\theta) - \sin(3\theta) = 2\sin\frac{5\theta-3\theta}{2}\cos\frac{5\theta+3\theta}{2}$$
$$= 2\sin\theta\cos(4\theta)$$

(b) $$\cos(3\theta) + \cos(2\theta) = 2\cos\frac{3\theta+2\theta}{2}\cos\frac{3\theta-2\theta}{2} \quad \text{Formula (8)}$$
$$= 2\cos\frac{5\theta}{2}\cos\frac{\theta}{2}$$

Now Work PROBLEM 11

3.6 Assess Your Understanding

Skill Building

In Problems 1–10, express each product as a sum containing only sines or only cosines.

1. $\sin(4\theta)\sin(2\theta)$
2. $\cos(4\theta)\cos(2\theta)$
3. $\sin(4\theta)\cos(2\theta)$
4. $\sin(3\theta)\sin(5\theta)$
5. $\cos(3\theta)\cos(5\theta)$
6. $\sin(4\theta)\cos(6\theta)$
7. $\sin\theta\sin(2\theta)$
8. $\cos(3\theta)\cos(4\theta)$
9. $\sin\frac{3\theta}{2}\cos\frac{\theta}{2}$
10. $\sin\frac{\theta}{2}\cos\frac{5\theta}{2}$

In Problems 11–18, express each sum or difference as a product of sines and/or cosines.

11. $\sin(4\theta) - \sin(2\theta)$
12. $\sin(4\theta) + \sin(2\theta)$
13. $\cos(2\theta) + \cos(4\theta)$
14. $\cos(5\theta) - \cos(3\theta)$
15. $\sin\theta + \sin(3\theta)$
16. $\cos\theta + \cos(3\theta)$
17. $\cos\frac{\theta}{2} - \cos\frac{3\theta}{2}$
18. $\sin\frac{\theta}{2} - \sin\frac{3\theta}{2}$

In Problems 19–36, establish each identity.

19. $\dfrac{\sin\theta + \sin(3\theta)}{2\sin(2\theta)} = \cos\theta$
20. $\dfrac{\cos\theta + \cos(3\theta)}{2\cos(2\theta)} = \cos\theta$
21. $\dfrac{\sin(4\theta) + \sin(2\theta)}{\cos(4\theta) + \cos(2\theta)} = \tan(3\theta)$
22. $\dfrac{\cos\theta - \cos(3\theta)}{\sin(3\theta) - \sin\theta} = \tan(2\theta)$
23. $\dfrac{\cos\theta - \cos(3\theta)}{\sin\theta + \sin(3\theta)} = \tan\theta$
24. $\dfrac{\cos\theta - \cos(5\theta)}{\sin\theta + \sin(5\theta)} = \tan(2\theta)$
25. $\sin\theta[\sin\theta + \sin(3\theta)] = \cos\theta[\cos\theta - \cos(3\theta)]$
26. $\sin\theta[\sin(3\theta) + \sin(5\theta)] = \cos\theta[\cos(3\theta) - \cos(5\theta)]$
27. $\dfrac{\sin(4\theta) + \sin(8\theta)}{\cos(4\theta) + \cos(8\theta)} = \tan(6\theta)$
28. $\dfrac{\sin(4\theta) - \sin(8\theta)}{\cos(4\theta) - \cos(8\theta)} = -\cot(6\theta)$
29. $\dfrac{\sin(4\theta) + \sin(8\theta)}{\sin(4\theta) - \sin(8\theta)} = -\dfrac{\tan(6\theta)}{\tan(2\theta)}$
30. $\dfrac{\cos(4\theta) - \cos(8\theta)}{\cos(4\theta) + \cos(8\theta)} = \tan(2\theta)\tan(6\theta)$
31. $\dfrac{\sin\alpha + \sin\beta}{\sin\alpha - \sin\beta} = \tan\dfrac{\alpha+\beta}{2}\cot\dfrac{\alpha-\beta}{2}$
32. $\dfrac{\cos\alpha + \cos\beta}{\cos\alpha - \cos\beta} = -\cot\dfrac{\alpha+\beta}{2}\cot\dfrac{\alpha-\beta}{2}$

33. $\dfrac{\sin\alpha + \sin\beta}{\cos\alpha + \cos\beta} = \tan\dfrac{\alpha + \beta}{2}$

34. $\dfrac{\sin\alpha - \sin\beta}{\cos\alpha - \cos\beta} = -\cot\dfrac{\alpha + \beta}{2}$

35. $1 + \cos(2\theta) + \cos(4\theta) + \cos(6\theta) = 4\cos\theta\cos(2\theta)\cos(3\theta)$

36. $1 - \cos(2\theta) + \cos(4\theta) - \cos(6\theta) = 4\sin\theta\cos(2\theta)\sin(3\theta)$

Applications and Extensions

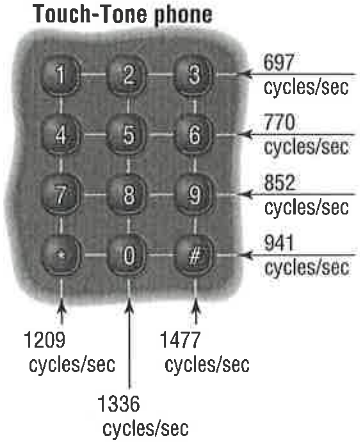
37. Touch-Tone Phones On a Touch-Tone phone, each button produces a unique sound. The sound produced is the sum of two tones, given by

$$y = \sin(2\pi lt) \quad \text{and} \quad y = \sin(2\pi ht)$$

where l and h are the low and high frequencies (cycles per second) shown on the illustration. For example, if you touch 7, the low frequency is $l = 852$ cycles per second and the high frequency is $h = 1209$ cycles per second. The sound emitted by touching 7 is

$$y = \sin[2\pi(852)t] + \sin[2\pi(1209)t]$$

Touch-Tone phone



(a) Write this sound as a product of sines and/or cosines.
(b) Determine the maximum value of y.
(c) Graph the sound emitted by touching 7.

38. Touch-Tone Phones
(a) Write the sound emitted by touching the # key as a product of sines and/or cosines.
(b) Determine the maximum value of y.
(c) Graph the sound emitted by touching the # key.

39. Moment of Inertia The moment of inertia I of an object is a measure of how easy it is to rotate the object about some fixed point. In engineering mechanics, it is sometimes necessary to compute moments of inertia with respect to a set of rotated axes. These moments are given by the equations

$$I_u = I_x\cos^2\theta + I_y\sin^2\theta - 2I_{xy}\sin\theta\cos\theta$$
$$I_v = I_x\sin^2\theta + I_y\cos^2\theta + 2I_{xy}\sin\theta\cos\theta$$

Use product-to-sum formulas to show that

$$I_u = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - I_{xy}\sin 2\theta$$

and

$$I_v = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2}\cos 2\theta + I_{xy}\sin 2\theta$$

Source: Adapted from Hibbeler, *Engineering Mechanics: Statics*, 10th ed., Prentice Hall © 2004.

40. Projectile Motion The range of a projectile propelled downward from the top of an inclined plane at an angle θ to the inclined plane is given by

$$R(\theta) = \frac{2v_0^2\sin\theta\cos(\theta - \phi)}{g\cos^2\phi}$$

when the projectile hits the inclined plane. Here v_0 is the initial velocity of the projectile, ϕ is the angle the plane makes with respect to the horizontal, and g is acceleration due to gravity.

(a) Show that for fixed v_0 and ϕ the maximum range down the incline is given by $R_{\max} = \dfrac{v_0^2}{g(1 - \sin\phi)}$.
(b) Determine the maximum range if the projectile has an initial velocity of 50 meters/second, the angle of the plane is $\phi = 35°$, and $g = 9.8$ meters/second^2.

41. If $\alpha + \beta + \gamma = \pi$, show that

$$\sin(2\alpha) + \sin(2\beta) + \sin(2\gamma) = 4\sin\alpha\sin\beta\sin\gamma$$

42. If $\alpha + \beta + \gamma = \pi$, show that

$$\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\tan\beta\tan\gamma$$

43. Derive formula (3).

44. Derive formula (7).

45. Derive formula (8).

46. Derive formula (9).

3.7 Trigonometric Equations (I)

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Solving Equations (Appendix A, Section A.4, pp. A25–A28)
- Values of the Trigonometric Functions (Section 2.2, pp. 112 and 115)

Now Work the 'Are You Prepared?' problems on page 229.

OBJECTIVE 1 Solve Equations Involving a Single Trigonometric Function (p. 226)

EXAMPLE 6 **Solving a Trigonometric Equation with a Calculator**

Use a calculator to solve the equation $\sin\theta = 0.3$, $0 \le \theta < 2\pi$. Express any solutions in radians, rounded to two decimal places.

Solution To solve $\sin\theta = 0.3$ on a calculator, first set the mode to radians. Then use the $\boxed{\sin^{-1}}$ key to obtain

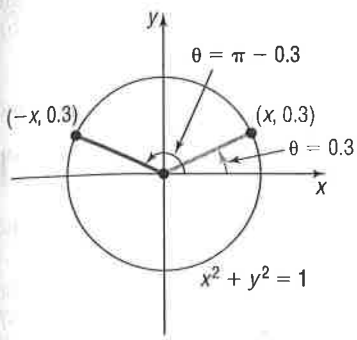
$$\theta = \sin^{-1}(0.3) \approx 0.3046927$$

Rounded to two decimal places, $\theta = \sin^{-1}(0.3) = 0.30$ radian. Because of the definition of $y = \sin^{-1} x$, the angle θ that we obtain is the angle $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ for which $\sin\theta = 0.3$. Another angle for which $\sin\theta = 0.3$ is $\pi - 0.30$. See Figure 29. The angle $\pi - 0.30$ is the angle in quadrant II, where $\sin\theta = 0.3$. The solutions for $\sin\theta = 0.3$, $0 \le \theta < 2\pi$, are

$$\theta = 0.30 \text{ radian} \quad \text{and} \quad \theta = \pi - 0.30 \approx 2.84 \text{ radians}$$

The solution set is $\{0.30, 2.84\}$.

Figure 29



WARNING Example 6 illustrates that caution must be exercised when solving trigonometric equations on a calculator. Remember that the calculator supplies an angle only within the restrictions of the definition of the inverse trigonometric function. To find the remaining solutions, you must identify other quadrants, if any, in which a solution may be located.

Now Work PROBLEM 41

3.7 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. Solve: $3x - 5 = -x + 1$ (pp. A25–A28)
2. $\sin\left(\frac{\pi}{4}\right) =$ _____; $\cos\left(\frac{8\pi}{3}\right) =$ _____ (pp. 112 and 115)

Concepts and Vocabulary

3. Two solutions of the equation $\sin\theta = \frac{1}{2}$ are _____ and _____.
4. All the solutions of the equation $\sin\theta = \frac{1}{2}$ are _____.
5. ***True or False*** Most trigonometric equations have unique solutions.
6. ***True or False*** The equation $\sin\theta = 2$ has a real solution that can be found using a calculator.

Skill Building

In Problems 7–30, solve each equation on the interval $0 \le \theta < 2\pi$.

7. $2\sin\theta + 3 = 2$
8. $1 - \cos\theta = \frac{1}{2}$
9. $4\cos^2\theta = 1$
10. $\tan^2\theta = \frac{1}{3}$
11. $2\sin^2\theta - 1 = 0$
12. $4\cos^2\theta - 3 = 0$
13. $\sin(3\theta) = -1$
14. $\tan\frac{\theta}{2} = \sqrt{3}$
15. $\cos(2\theta) = -\frac{1}{2}$
16. $\tan(2\theta) = -1$
17. $\sec\frac{3\theta}{2} = -2$
18. $\cot\frac{2\theta}{3} = -\sqrt{3}$
19. $2\sin\theta + 1 = 0$
20. $\cos\theta + 1 = 0$
21. $\tan\theta + 1 = 0$
22. $\sqrt{3}\cot\theta + 1 = 0$
23. $4\sec\theta + 6 = -2$
24. $5\csc\theta - 3 = 2$
25. $3\sqrt{2}\cos\theta + 2 = -1$
26. $4\sin\theta + 3\sqrt{3} = \sqrt{3}$
27. $\cos\left(2\theta - \frac{\pi}{2}\right) = -1$
28. $\sin\left(3\theta + \frac{\pi}{18}\right) = 1$
29. $\tan\left(\frac{\theta}{2} + \frac{\pi}{3}\right) = 1$
30. $\cos\left(\frac{\theta}{3} - \frac{\pi}{4}\right) = \frac{1}{2}$

In Problems 31–40, solve each equation. Give a general formula for all the solutions. List six solutions.

31. $\sin\theta = \frac{1}{2}$
32. $\tan\theta = 1$
33. $\tan\theta = -\frac{\sqrt{3}}{3}$
34. $\cos\theta = -\frac{\sqrt{3}}{2}$
35. $\cos\theta = 0$

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36. $\sin\theta = \frac{\sqrt{2}}{2}$ **37.** $\cos(2\theta) = -\frac{1}{2}$ **38.** $\sin(2\theta) = -1$ **39.** $\sin\frac{\theta}{2} = -\frac{\sqrt{3}}{2}$ **40.** $\tan\frac{\theta}{2} = -1$

In Problems 41–52, use a calculator to solve each equation on the interval $0 \le \theta < 2\pi$. *Round answers to two decimal places.*

41. $\sin\theta = 0.4$ **42.** $\cos\theta = 0.6$ **43.** $\tan\theta = 5$ **44.** $\cot\theta = 2$

45. $\cos\theta = -0.9$ **46.** $\sin\theta = -0.2$ **47.** $\sec\theta = -4$ **48.** $\csc\theta = -3$

49. $5\tan\theta + 9 = 0$ **50.** $4\cot\theta = -5$ **51.** $3\sin\theta - 2 = 0$ **52.** $4\cos\theta + 3 = 0$

Applications and Extensions

53. What are the x-intercepts of the graph of $f(x) = 4\sin^2 x - 3$ on the interval $[0, 2\pi]$?

54. What are the x-intercepts of the graph of $f(x) = 2\cos(3x) + 1$ on the interval $[0, \pi]$?

55. $f(x) = 3\sin x$.
(a) Find the x-intercepts of the graph of f on the interval $[-2\pi, 4\pi]$.
(b) Graph $f(x) = 3\sin x$ on the interval $[-2\pi, 4\pi]$.
(c) Solve $f(x) = \frac{3}{2}$ on the interval $[-2\pi, 4\pi]$. What points are on the graph of f? Label these points on the graph drawn in part (b).
(d) Use the graph drawn in part (b) along with the results of part (c) to determine the values of x such that $f(x) > \frac{3}{2}$ on the interval $[-2\pi, 4\pi]$.

56. $f(x) = 2\cos x$.
(a) Find the x-intercepts of the graph of f on the interval $[-2\pi, 4\pi]$.
(b) Graph $f(x) = 2\cos x$ on the interval $[-2\pi, 4\pi]$.
(c) Solve $f(x) = -\sqrt{3}$ on the interval $[-2\pi, 4\pi]$. What points are on the graph of f? Label these points on the graph drawn in part (b).
(d) Use the graph drawn in part (b) along with the results of part (c) to determine the values of x such that $f(x) < -\sqrt{3}$ on the interval $[-2\pi, 4\pi]$.

57. $f(x) = 4\tan x$.
(a) Solve $f(x) = -4$.
(b) For what values of x is $f(x) < -4$ on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$?

58. $f(x) = \cot x$.
(a) Solve $f(x) = -\sqrt{3}$.
(b) For what values of x is $f(x) > -\sqrt{3}$ on the interval $(0, \pi)$?

59. (a) Graph $f(x) = 3\sin(2x) + 2$ and $g(x) = \frac{7}{2}$ on the same Cartesian plane for the interval $[0, \pi]$.
(b) Solve $f(x) = g(x)$ on the interval $[0, \pi]$ and label the points of intersection on the graph drawn in part (b).
(c) Solve $f(x) > g(x)$ on the interval $[0, \pi]$.
(d) Shade the region bounded by $f(x) = 3\sin(2x) + 2$ and $g(x) = \frac{7}{2}$ between the two points found in part (b) on the graph drawn in part (a).

60. (a) Graph $f(x) = 2\cos\frac{x}{2} + 3$ and $g(x) = 4$ on the same Cartesian plane for the interval $[0, 4\pi]$.
(b) Solve $f(x) = g(x)$ on the interval $[0, 4\pi]$ and label the points of intersection on the graph drawn in part (b).
(c) Solve $f(x) < g(x)$ on the interval $[0, 4\pi]$.
(d) Shade the region bounded by $f(x) = 2\cos\frac{x}{2} + 3$ and $g(x) = 4$ between the two points found in part (b) on the graph drawn in part (a).

61. (a) Graph $f(x) = -4\cos x$ and $g(x) = 2\cos x + 3$ on the same Cartesian plane for the interval $[0, 2\pi]$.
(b) Solve $f(x) = g(x)$ on the interval $[0, 2\pi]$ and label the points of intersection on the graph drawn in part (b).
(c) Solve $f(x) > g(x)$ on the interval $[0, 2\pi]$.
(d) Shade the region bounded by $f(x) = -4\cos x$ and $g(x) = 2\cos x + 3$ between the two points found in part (b) on the graph drawn in part (a).

62. (a) Graph $f(x) = 2\sin x$ and $g(x) = -2\sin x + 2$ on the same Cartesian plane for the interval $[0, 2\pi]$.
(b) Solve $f(x) = g(x)$ on the interval $[0, 2\pi]$ and label the points of intersection on the graph drawn in part (b).
(c) Solve $f(x) > g(x)$ on the interval $[0, 2\pi]$.
(d) Shade the region bounded by $f(x) = 2\sin x$ and $g(x) = -2\sin x + 2$ between the two points found in part (b) on the graph drawn in part (a).

63. The Ferris Wheel In 1893, George Ferris engineered the Ferris Wheel. It was 250 feet in diameter. If the wheel makes 1 revolution every 40 seconds, then the function

$$h(t) = 125\sin\left(0.157t - \frac{\pi}{2}\right) + 125$$

represents the height h, in feet, of a seat on the wheel as a function of time t, where t is measured in seconds. The ride begins when $t = 0$.
(a) During the first 40 seconds of the ride, at what time t is an individual on the Ferris Wheel exactly 125 feet above the ground?
(b) During the first 80 seconds of the ride, at what time t is an individual on the Ferris Wheel exactly 250 feet above the ground?
(c) During the first 40 seconds of the ride, over what interval of time t is an individual on the Ferris Wheel more than 125 feet above the ground?

64. Tire Rotation The P215/65R15 Cobra Radial G/T tire has a diameter of exactly 26 inches. Suppose that a car wheel is making 2 revolutions per second (the car is traveling a little less than 10 miles per hour). Then $h(t) = 13\sin\left(4\pi t - \frac{\pi}{2}\right) + 13$ represents the height h (in inches) of a point on the tire as a function of time t (in seconds). The car starts to move when $t = 0$.

(a) During the first second that the car is moving, at what time t is the point on the tire exactly 13 inches above the ground?

(b) During the first second that the car is moving, at what time t is the point on the tire exactly 6.5 inches above the ground?

(c) During the first second that the car is moving, at what time t is the point on the tire more than 13 inches above the ground?

Source: *Cobra Tire*

65. Holding Pattern An airplane is asked to stay within a holding pattern near Chicago's O'Hare International Airport. The function $d(x) = 70\sin(0.65x) + 150$ represents the distance d, in miles, that the airplane is from the airport at time x, in minutes.

(a) When the plane enters the holding pattern, $x = 0$, how far is it from O'Hare?

(b) During the first 20 minutes after the plane enters the holding pattern, at what time x is the plane exactly 100 miles from the airport?

(c) During the first 20 minutes after the plane enters the holding pattern, at what time x is the plane more than 100 miles from the airport?

(d) While the plane is in the holding pattern, will it ever be within 70 miles of the airport? Why?

66. Projectile Motion A golfer hits a golf ball with an initial velocity of 100 miles per hour. The range R of the ball as a function of the angle θ to the horizontal is given by $R(\theta) = 672\sin(2\theta)$, where R is measured in feet.

(a) At what angle θ should the ball be hit if the golfer wants the ball to travel 450 feet (150 yards)?

(b) At what angle θ should the ball be hit if the golfer wants the ball to travel 540 feet (180 yards)?

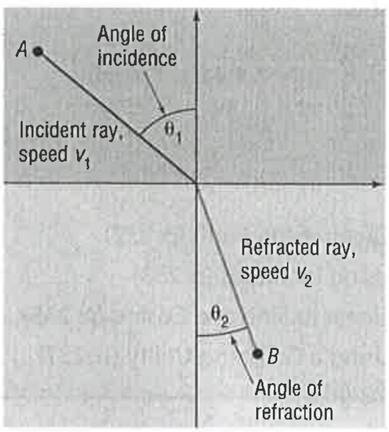
(c) At what angle θ should the ball be hit if the golfer wants the ball to travel at least 480 feet (160 yards)?

(d) Can the golfer hit the ball 720 feet (240 yards)?

The following discussion of Snell's Law of Refraction (named after Willebrord Snell, 1580–1626) is needed for Problems 67–73. Light, sound, and other waves travel at different speeds, depending on the media (air, water, wood, and so on) through which they pass. Suppose that light travels from a point A in one medium, where its speed is v_1, to a point B in another medium, where its speed is v_2. Refer to the figure, where the angle θ_1 is called the* ***angle of incidence*** *and the angle θ_2 is the* ***angle of refraction.*** *Snell's Law, which can be proved using calculus, states that*

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}$$

The ratio $\frac{v_1}{v_2}$ is called the ***index of refraction.*** *Some values are given in the following table.*



Some Indexes of Refraction

Medium	Index of Refraction†
Water	1.33
Ethyl alcohol (20°C)	1.36
Carbon disulfide	1.63
Air (1 atm and 0°C)	1.00029
Diamond	2.42
Fused quartz	1.46
Glass, crown	1.52
Glass, dense flint	1.66
Sodium chloride	1.54

67. The index of refraction of light in passing from a vacuum into water is 1.33. If the angle of incidence is 40°, determine the angle of refraction.

68. The index of refraction of light in passing from a vacuum into dense flint glass is 1.66. If the angle of incidence is 50°, determine the angle of refraction.

69. Ptolemy, who lived in the city of Alexandria in Egypt during the second century AD, gave the measured values in the table following for the angle of incidence θ_1 and the angle of refraction θ_2 for a light beam passing from air into water. Do these values agree with Snell's Law? If so, what index of refraction results? (These data are interesting as the oldest recorded physical measurements.)†

θ_1	θ_2	θ_1	θ_2
10°	8°	50°	35°0′
20°	15°30′	60°	40°30′
30°	22°30′	70°	45°30′
40°	29°0′	80°	50°0′

* Because this law was also deduced by René Descartes in France, it is also known as Descartes's Law.

† For light of wavelength 589 nanometers, measured with respect to a vacuum. The index with respect to air is negligibly different in most cases.

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70. **Bending Light** The speed of yellow sodium light (wavelength of 589 nanometers) in a certain liquid is measured to be 1.92×10^8 meters per second. What is the index of refraction of this liquid, with respect to air, for sodium light?*

[**Hint:** The speed of light in air is approximately 2.998×10^8 meters per second.]

71. **Bending Light** A beam of light with a wavelength of 589 nanometers traveling in air makes an angle of incidence of 40° on a slab of transparent material, and the refracted beam makes an angle of refraction of 26°. Find the index of refraction of the material.*

72. **Bending Light** A light ray with a wavelength of 589 nanometers (produced by a sodium lamp) traveling through air makes an angle of incidence of 30° on a smooth, flat slab of crown glass. Find the angle of refraction.*

73. A light beam passes through a thick slab of material wh index of refraction is n_2. Show that the emerging beam is p allel to the incident beam.*

74. **Brewster's Law** If the angle of incidence and the angl refraction are complementary angles, the angle of incide is referred to as the Brewster angle θ_B. The Brewster ar is related to the index of refractions of the two media and n_2, by the equation $n_1 \sin \theta_B = n_2 \cos \theta_B$, where n the index of refraction of the incident medium and n_2 is index of refraction of the refractive medium. Determ the Brewster angle for a light beam traveling through wa (at 20°C) that makes an angle of incidence with a smo flat slab of crown glass.

* Adapted from Halliday and Resnick, *Fundamentals of Physics,* 7th ed., 2005, John Wiley & Sons.

Discussion and Writing

75. Explain in your own words how you would use your calculator to solve the equation $\cos x = -0.6, 0 \le x < 2\pi$. How would modify your approach to solve the equation $\cot x = 5, 0 < x < 2\pi$?

'Are You Prepared?' Answers

1. $\left\{\frac{3}{2}\right\}$ 2. $\frac{\sqrt{2}}{2}; -\frac{1}{2}$

3.8 Trigonometric Equations (II)

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Solving Quadratic Equations by Factoring (Appendix A, Section A.4, pp. A28–A29)
- The Quadratic Formula (Appendix A, Section A.4, pp. A31–A33)
- Using a Graphing Utility to Solve Equations (Appendix B, Section B.4, pp. B8–B10)

Now Work the 'Are You Prepared?' problems on page 237.

OBJECTIVES
1 Solve Trigonometric Equations Quadratic in Form (p. 232)
2 Solve Trigonometric Equations Using Identities (p. 233)
3 Solve Trigonometric Equations Linear in Sine and Cosine (p. 235)
4 Solve Trigonometric Equations Using a Graphing Utility (p. 237)

1 Solve Trigonometric Equations Quadratic in Form

In this section we continue our study of trigonometric equations. Many trigonom ric equations can be solved by applying techniques that we already know, such applying the quadratic formula (if the equation is a second-degree polynomial) factoring.

EXAMPLE 1 **Solving a Trigonometric Equation Quadratic in Form**

Solve the equation: $2\sin^2\theta - 3\sin\theta + 1 = 0, \quad 0 \le \theta < 2\pi$

Solution This equation is a quadratic equation (in $\sin\theta$) that can be factored.

$$2\sin^2\theta - 3\sin\theta + 1 = 0 \qquad 2x^2 - 3x + 1 = 0, \quad x = \sin\theta$$
$$(2\sin\theta - 1)(\sin\theta - 1) = 0 \qquad (2x - 1)(x - 1) = 0$$

If $|c| \le \sqrt{a^2 + b^2}$, then all the solutions of equation (5) are

$$\theta + \phi = \sin^{-1}\frac{c}{\sqrt{a^2 + b^2}} + 2k\pi \quad \text{or} \quad \theta + \phi = \pi - \sin^{-1}\frac{c}{\sqrt{a^2 + b^2}} + 2k\pi$$

Because the angle ϕ is determined by equations (4), these provide the solutions to equation (2).

Now Work PROBLEM 41

4 Solve Trigonometric Equations Using a Graphing Utility

The techniques introduced in this section apply only to certain types of trigonometric equations. Solutions for other types are usually studied in calculus, using numerical methods. In the next example, we show how a graphing utility may be used to obtain solutions.

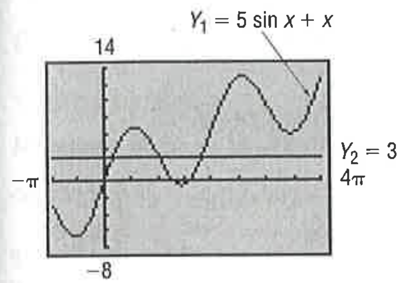
EXAMPLE 8 **Solving Trigonometric Equations Using a Graphing Utility**

Solve: $5 \sin x + x = 3$

Express the solution(s) rounded to two decimal places.

Solution This type of trigonometric equation cannot be solved by previous methods. A graphing utility, though, can be used here. The solution(s) of this equation is the same as the points of intersection of the graphs of $Y_1 = 5 \sin x + x$ and $Y_2 = 3$. See Figure 32.

Figure 32



There are three points of intersection; the x-coordinates are the solutions that we seek. Using INTERSECT, we find

$$x = 0.52, \qquad x = 3.18, \qquad x = 5.71$$

The solution set is $\{0.52, 3.18, 5.71\}$.

Now Work PROBLEM 53

3.8 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. Find the real solutions of $4x^2 - x - 5 = 0$. (pp. A28–A29)
2. Find the real solutions of $x^2 - x - 1 = 0$. (pp. A31–A33)
3. Find the real solutions of $(2x - 1)^2 - 3(2x - 1) - 4 = 0$. (pp. A31–A33)
4. Use a graphing utility to solve $5x^3 - 2 = x - x^2$. Round answers to two decimal places. (pp. B8–B10)

Skill Building

In Problems 5–46, solve each equation on the interval $0 \le \theta < 2\pi$.

5. $2\cos^2\theta + \cos\theta = 0$
6. $\sin^2\theta - 1 = 0$
7. $2\sin^2\theta - \sin\theta - 1 = 0$
8. $2\cos^2\theta + \cos\theta - 1 = 0$
9. $(\tan\theta - 1)(\sec\theta - 1) = 0$
10. $(\cot\theta + 1)\left(\csc\theta - \frac{1}{2}\right) = 0$
11. $\sin^2\theta - \cos^2\theta = 1 + \cos\theta$
12. $\cos^2\theta - \sin^2\theta + \sin\theta = 0$
13. $\sin^2\theta = 6(\cos\theta + 1)$
14. $2\sin^2\theta = 3(1 - \cos\theta)$
15. $\cos(2\theta) + 6\sin^2\theta = 4$
16. $\cos(2\theta) = 2 - 2\sin^2\theta$
17. $\cos\theta = \sin\theta$
18. $\cos\theta + \sin\theta = 0$
19. $\tan\theta = 2\sin\theta$
20. $\sin(2\theta) = \cos\theta$
21. $\sin\theta = \csc\theta$
22. $\tan\theta = \cot\theta$
23. $\cos(2\theta) = \cos\theta$
24. $\sin(2\theta)\sin\theta = \cos\theta$
25. $\sin(2\theta) + \sin(4\theta) = 0$
26. $\cos(2\theta) + \cos(4\theta) = 0$
27. $\cos(4\theta) - \cos(6\theta) = 0$
28. $\sin(4\theta) - \sin(6\theta) = 0$

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29. $1 + \sin\theta = 2\cos^2\theta$

30. $\sin^2\theta = 2\cos\theta + 2$

31. $2\sin^2\theta - 5\sin\theta + 3 = 0$

32. $2\cos^2\theta - 7\cos\theta - 4 = 0$

33. $3(1 - \cos\theta) = \sin^2\theta$

34. $4(1 + \sin\theta) = \cos^2\theta$

35. $\tan^2\theta = \frac{3}{2}\sec\theta$

36. $\csc^2\theta = \cot\theta + 1$

37. $3 - \sin\theta = \cos(2\theta)$

38. $\cos(2\theta) + 5\cos\theta + 3 = 0$

39. $\sec^2\theta + \tan\theta = 0$

40. $\sec\theta = \tan\theta + \cot\theta$

41. $\sin\theta - \sqrt{3}\cos\theta = 1$

42. $\sqrt{3}\sin\theta + \cos\theta = 1$

43. $\tan(2\theta) + 2\sin\theta = 0$

44. $\tan(2\theta) + 2\cos\theta = 0$

45. $\sin\theta + \cos\theta = \sqrt{2}$

46. $\sin\theta + \cos\theta = -\sqrt{2}$

In Problems 47–52, find the x-intercepts of the graph of each trigonometric function on the interval $0 \le x < 2\pi$.

47. $f(x) = 4\cos^2 x - 1$

48. $f(x) = 4\sin^2 x - 3$

49. $f(x) = \sin(2x) - \sin x$

50. $f(x) = \cos(2x) + \cos x$

51. $f(x) = \sin^2 x + 2\cos x + 2$

52. $f(x) = \cos(2x) + \sin^2 x$

In Problems 53–64, use a graphing utility to solve each equation. Express the solution(s) rounded to two decimal places.

53. $x + 5\cos x = 0$

54. $x - 4\sin x = 0$

55. $22x - 17\sin x = 3$

56. $19x + 8\cos x = 2$

57. $\sin x + \cos x = x$

58. $\sin x - \cos x = x$

59. $x^2 - 2\cos x = 0$

60. $x^2 + 3\sin x = 0$

61. $x^2 - 2\sin(2x) = 3x$

62. $x^2 = x + 3\cos(2x)$

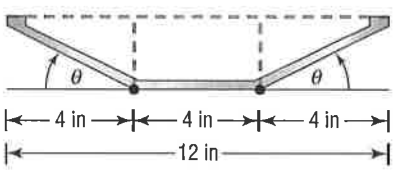
63. $6\sin x - e^x = 2, \quad x > 0$

64. $4\cos(3x) - e^x = 1, \quad x > 0$

Applications and Extensions

65. Constructing a Rain Gutter A rain gutter is to be constructed of aluminum sheets 12 inches wide. After marking off a length of 4 inches from each edge, this length is bent up at an angle θ. See the illustration. The area A of the opening as a function of θ is given by

$$A(\theta) = 16\sin\theta(\cos\theta + 1) \quad 0° < \theta < 90°$$



(a) In calculus, you will be asked to find the angle θ that maximizes A by solving the equation

$$\cos(2\theta) + \cos\theta = 0, \quad 0° < \theta < 90°$$

Solve this equation for θ by using the Double-angle Formula.

(b) Solve the equation in part (a) for θ by writing the sum of the two cosines as a product.

(c) What is the maximum area A of the opening?

(d) Graph $A = A(\theta), 0° \le \theta \le 90°$, and find the angle θ that maximizes the area A. Also find the maximum area. Compare the results to the answers found earlier.

66. Projectile Motion An object is propelled upward at an angle θ, $45° < \theta < 90°$, to the horizontal with an initial velocity of v_0 feet per second from the base of an inclined plane that makes an angle of 45° with the horizontal. See the illustration. If air resistance is ignored, the distance R that the object travels up the inclined plane is given by

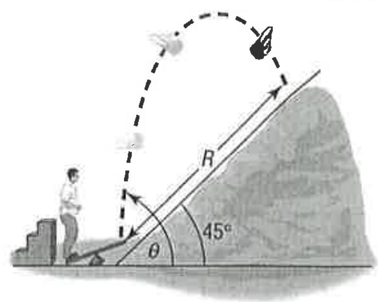
$$R(\theta) = \frac{v_0^2\sqrt{2}}{32}[\sin(2\theta) - \cos(2\theta) - 1]$$

(a) In calculus, you will be asked to find the angle θ that maximizes R by solving the equation

$$\sin(2\theta) + \cos(2\theta) = 0$$

Solve this equation for θ using the method of Example 7.

(b) Solve this equation for θ by dividing each side by $\cos(2\theta)$.



(c) What is the maximum distance R if $v_0 = 32$ feet per s ond?

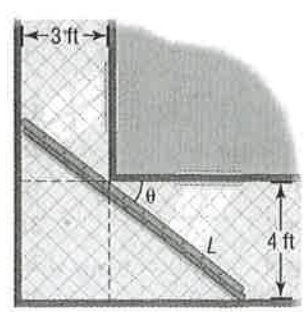
(d) Graph $R = R(\theta), 45° \le \theta \le 90°$, and find the angl that maximizes the distance R. Also find the maximu distance. Use $v_0 = 32$ feet per second. Compare the sults with the answers found earlier.

67. Heat Transfer In the study of heat transfer, the equati $x + \tan x = 0$ occurs. Graph $Y_1 = -x$ and $Y_2 = \tan x$ $x \ge 0$. Conclude that there are an infinite number of poi of intersection of these two graphs. Now find the first t positive solutions of $x + \tan x = 0$ rounded to two decin places.

68. Carrying a Ladder Around a Corner Two hallways, one width 3 feet, the other of width 4 feet, meet at a right ang See the illustration.

(a) Show that the length L as a function of θ is

$$L(\theta) = 4\csc\theta + 3\sec\theta$$



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(b) In calculus, you will be asked to find the length of the longest ladder that can turn the corner by solving the equation

$$3 \sec \theta \tan \theta - 4 \csc \theta \cot \theta = 0, \quad 0° < \theta < 90°$$

Solve this equation for θ.

(c) What is the length of the longest ladder that can be carried around the corner?

(d) Graph $L = L(\theta), 0° \leq \theta \leq 90°$, and find the angle θ that minimizes the length L.

(e) Compare the result with the one found in part (b). Explain why the two answers are the same.

69. **Projectile Motion** The horizontal distance that a projectile will travel in the air (ignoring air resistance) is given by the equation

$$R(\theta) = \frac{v_0^2 \sin(2\theta)}{g}$$

where v_0 is the initial velocity of the projectile, θ is the angle of elevation, and g is acceleration due to gravity (9.8 meters per second squared).

(a) If you can throw a baseball with an initial speed of 34.8 meters per second, at what angle of elevation θ should you direct the throw so that the ball travels a distance of 107 meters before striking the ground?

(b) Determine the maximum distance that you can throw the ball.

(c) Graph $R = R(\theta)$, with $v_0 = 34.8$ meters per second.

(d) Verify the results obtained in parts (a) and (b) using a graphing utility.

70. **Projectile Motion** Refer to Problem 69.

(a) If you can throw a baseball with an initial speed of 40 meters per second, at what angle of elevation θ should you direct the throw so that the ball travels a distance of 110 meters before striking the ground?

(b) Determine the maximum distance that you can throw the ball.

(c) Graph $R = R(\theta)$, with $v_0 = 40$ meters per second.

(d) Verify the results obtained in parts (a) and (b) using a graphing utility.

'Are You Prepared?' Answers

1. $\left\{-1, \frac{5}{4}\right\}$ 2. $\left\{\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right\}$ 3. $\left\{0, \frac{5}{2}\right\}$ 4. $\{0.76\}$

CHAPTER REVIEW

Things to Know

Definitions of the six inverse trigonometric functions

$y = \sin^{-1} x$ means $x = \sin y$ where $-1 \leq x \leq 1, \quad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ (p. 179)

$y = \cos^{-1} x$ means $x = \cos y$ where $-1 \leq x \leq 1, \quad 0 \leq y \leq \pi$ (p. 182)

$y = \tan^{-1} x$ means $x = \tan y$ where $-\infty < x < \infty, \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$ (p. 185)

$y = \sec^{-1} x$ means $x = \sec y$ where $|x| \geq 1, \quad 0 \leq y \leq \pi, \quad y \neq \frac{\pi}{2}$ (p. 192)

$y = \csc^{-1} x$ means $x = \csc y$ where $|x| \geq 1, \quad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, \quad y \neq 0$ (p. 192)

$y = \cot^{-1} x$ means $x = \cot y$ where $-\infty < x < \infty, \quad 0 < y < \pi$ (p. 192)

Sum and Difference Formulas (pp. 203, 206, and 208)

$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$ $\quad$ $\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$

$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$ $\quad$ $\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$

$\tan(\alpha + \beta) = \dfrac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$ $\quad$ $\tan(\alpha - \beta) = \dfrac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$

Double-angle Formulas (pp. 213 and 214)

$\sin(2\theta) = 2 \sin \theta \cos \theta$ $\quad$ $\cos(2\theta) = \cos^2 \theta - \sin^2 \theta$ $\quad$ $\tan(2\theta) = \dfrac{2 \tan \theta}{1 - \tan^2 \theta}$

$\cos(2\theta) = 2 \cos^2 \theta - 1$ $\quad$ $\cos(2\theta) = 1 - 2 \sin^2 \theta$

Half-angle Formulas (pp. 216, 217 and 219)

$\sin^2 \frac{\alpha}{2} = \dfrac{1 - \cos \alpha}{2}$ $\quad$ $\cos^2 \frac{\alpha}{2} = \dfrac{1 + \cos \alpha}{2}$ $\quad$ $\tan^2 \frac{\alpha}{2} = \dfrac{1 - \cos \alpha}{1 + \cos \alpha}$

$\sin \frac{\alpha}{2} = \pm\sqrt{\dfrac{1 - \cos \alpha}{2}}$ $\quad$ $\cos \frac{\alpha}{2} = \pm\sqrt{\dfrac{1 + \cos \alpha}{2}}$ $\quad$ $\tan \frac{\alpha}{2} = \pm\sqrt{\dfrac{1 - \cos \alpha}{1 + \cos \alpha}} = \dfrac{1 - \cos \alpha}{\sin \alpha} = \dfrac{\sin \alpha}{1 + \cos \alpha}$

where the + or − is determined by the quadrant of $\frac{\alpha}{2}$.