1. Given that

$$\lim_{x \to 2} f(x) = 4 \qquad \lim_{x \to 2} g(x) = -2 \qquad \lim_{x \to 2} h(x) = 0$$

find the limits that exist. If the limit does not exist, explain why.

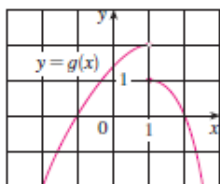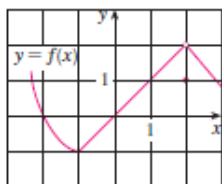(a) $\lim_{x \to 2} [f(x) + 5g(x)]$

(b) $\lim_{x \to 2} [g(x)]^3$

(c) $\lim_{x \to 2} \sqrt{f(x)}$

(d) $\lim_{x \to 2} \dfrac{3f(x)}{g(x)}$

(e) $\lim_{x \to 2} \dfrac{g(x)}{h(x)}$

(f) $\lim_{x \to 2} \dfrac{g(x)h(x)}{f(x)}$

2. The graphs of f and g are given. Use them to evaluate each limit, if it exists. If the limit does not exist, explain why.

(a) $\lim_{x \to 2} [f(x) + g(x)]$

(b) $\lim_{x \to 1} [f(x) + g(x)]$

(c) $\lim_{x \to 0} [f(x)g(x)]$

(d) $\lim_{x \to -1} \dfrac{f(x)}{g(x)}$

(e) $\lim_{x \to 2} [x^3 f(x)]$

(f) $\lim_{x \to 1} \sqrt{3 + f(x)}$

3–9 Evaluate the limit and justify each step by indicating the appropriate Limit Law(s).

3. $\lim_{x \to 3} (5x^3 - 3x^2 + x - 6)$

4. $\lim_{x \to -1} (x^4 - 3x)(x^2 + 5x + 3)$

5. $\lim_{t \to -2} \dfrac{t^4 - 2}{2t^2 - 3t + 2}$

6. $\lim_{u \to -2} \sqrt{u^4 + 3u + 6}$

7. $\lim_{x \to 8} (1 + \sqrt[3]{x})(2 - 6x^2 + x^3)$

8. $\lim_{t \to 2} \left(\dfrac{t^2 - 2}{t^3 - 3t + 5} \right)^2$

9. $\lim_{x \to 2} \sqrt{\dfrac{2x^2 + 1}{3x - 2}}$

10. (a) What is wrong with the following equation?

$$\frac{x^2 + x - 6}{x - 2} = x + 3$$

(b) In view of part (a), explain why the equation

$$\lim_{x \to 2} \frac{x^2 + x - 6}{x - 2} = \lim_{x \to 2} (x + 3)$$

is correct.

11–32 Evaluate the limit, if it exists.

11. $\lim\limits_{x \to 5} \dfrac{x^2 - 6x + 5}{x - 5}$

12. $\lim\limits_{x \to 4} \dfrac{x^2 - 4x}{x^2 - 3x - 4}$

13. $\lim\limits_{x \to 5} \dfrac{x^2 - 5x + 6}{x - 5}$

14. $\lim\limits_{x \to -1} \dfrac{x^2 - 4x}{x^2 - 3x - 4}$

15. $\lim\limits_{t \to -3} \dfrac{t^2 - 9}{2t^2 + 7t + 3}$

16. $\lim\limits_{x \to -1} \dfrac{2x^2 + 3x + 1}{x^2 - 2x - 3}$

17. $\lim\limits_{h \to 0} \dfrac{(-5 + h)^2 - 25}{h}$

18. $\lim\limits_{h \to 0} \dfrac{(2 + h)^3 - 8}{h}$

19. $\lim\limits_{x \to -2} \dfrac{x + 2}{x^3 + 8}$

20. $\lim\limits_{t \to 1} \dfrac{t^4 - 1}{t^3 - 1}$

21. $\lim\limits_{h \to 0} \dfrac{\sqrt{9 + h} - 3}{h}$

22. $\lim\limits_{u \to 2} \dfrac{\sqrt{4u + 1} - 3}{u - 2}$

23. $\lim\limits_{x \to -4} \dfrac{\frac{1}{4} + \frac{1}{x}}{4 + x}$

24. $\lim\limits_{x \to -1} \dfrac{x^2 + 2x + 1}{x^4 - 1}$

25. $\lim\limits_{t \to 0} \dfrac{\sqrt{1 + t} - \sqrt{1 - t}}{t}$

26. $\lim\limits_{t \to 0} \left(\dfrac{1}{t} - \dfrac{1}{t^2 + t} \right)$

27. $\lim\limits_{x \to 16} \dfrac{4 - \sqrt{x}}{16x - x^2}$

28. $\lim\limits_{h \to 0} \dfrac{(3 + h)^{-1} - 3^{-1}}{h}$

29. $\lim\limits_{t \to 0} \left(\dfrac{1}{t\sqrt{1 + t}} - \dfrac{1}{t} \right)$

30. $\lim\limits_{x \to -4} \dfrac{\sqrt{x^2 + 9} - 5}{x + 4}$

31. $\lim\limits_{h \to 0} \dfrac{(x + h)^3 - x^3}{h}$

32. $\lim\limits_{h \to 0} \dfrac{\frac{1}{(x + h)^2} - \frac{1}{x^2}}{h}$

33. (a) Estimate the value of
$$\lim_{x \to 0} \dfrac{x}{\sqrt{1 + 3x} - 1}$$
by graphing the function $f(x) = x/(\sqrt{1 + 3x} - 1)$.
(b) Make a table of values of $f(x)$ for x close to 0 and guess the value of the limit.
(c) Use the Limit Laws to prove that your guess is correct.

34. (a) Use a graph of
$$f(x) = \dfrac{\sqrt{3 + x} - \sqrt{3}}{x}$$
to estimate the value of $\lim_{x \to 0} f(x)$ to two decimal places.
(b) Use a table of values of $f(x)$ to estimate the limit to four decimal places.
(c) Use the Limit Laws to find the exact value of the limit.

35. Use the Squeeze Theorem to show that $\lim_{x \to 0} (x^2 \cos 20\pi x) = 0$. Illustrate by graphing the functions $f(x) = -x^2$, $g(x) = x^2 \cos 20\pi x$, and $h(x) = x^2$ on the same screen.

36. Use the Squeeze Theorem to show that
$$\lim_{x \to 0} \sqrt{x^3 + x^2}\, \sin \dfrac{\pi}{x} = 0$$
Illustrate by graphing the functions f, g, and h (in the notation of the Squeeze Theorem) on the same screen.

37. If $4x - 9 \le f(x) \le x^2 - 4x + 7$ for $x \ge 0$, find $\lim_{x \to 4} f(x)$.

38. If $2x \le g(x) \le x^4 - x^2 + 2$ for all x, evaluate $\lim_{x \to 1} g(x)$.

39. Prove that $\lim_{x \to 0} x^4 \cos \dfrac{2}{x} = 0$.

40. Prove that $\lim_{x \to 0^+} \sqrt{x}\,[1 + \sin^2(2\pi/x)] = 0$.

41–46 Find the limit, if it exists. If the limit does not exist, explain why.

41. $\lim\limits_{x \to 3} (2x + |x - 3|)$

42. $\lim\limits_{x \to -6} \dfrac{2x + 12}{|x + 6|}$

43. $\lim\limits_{x \to 0.5^-} \dfrac{2x - 1}{|2x^3 - x^2|}$

44. $\lim\limits_{x \to -2} \dfrac{2 - |x|}{2 + x}$

45. $\lim\limits_{x \to 0^-} \left(\dfrac{1}{x} - \dfrac{1}{|x|} \right)$

46. $\lim\limits_{x \to 0^+} \left(\dfrac{1}{x} - \dfrac{1}{|x|} \right)$

47. The *signum* (or sign) *function*, denoted by sgn, is defined by
$$\operatorname{sgn} x = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$
(a) Sketch the graph of this function.
(b) Find each of the following limits or explain why it does not exist.
  (i) $\lim\limits_{x \to 0^+} \operatorname{sgn} x$ (ii) $\lim\limits_{x \to 0^-} \operatorname{sgn} x$
  (iii) $\lim\limits_{x \to 0} \operatorname{sgn} x$ (iv) $\lim\limits_{x \to 0} |\operatorname{sgn} x|$

48. Let
$$f(x) = \begin{cases} x^2 + 1 & \text{if } x < 1 \\ (x - 2)^2 & \text{if } x \ge 1 \end{cases}$$
(a) Find $\lim_{x \to 1^-} f(x)$ and $\lim_{x \to 1^+} f(x)$.
(b) Does $\lim_{x \to 1} f(x)$ exist?
(c) Sketch the graph of f.

49. Let $g(x) = \dfrac{x^2 + x - 6}{|x - 2|}$.

(a) Find

   (i) $\lim\limits_{x \to 2^+} g(x)$ (ii) $\lim\limits_{x \to 2^-} g(x)$

(b) Does $\lim\limits_{x \to 2} g(x)$ exist?

(c) Sketch the graph of g.

50. Let

$$g(x) = \begin{cases} x & \text{if } x < 1 \\ 3 & \text{if } x = 1 \\ 2 - x^2 & \text{if } 1 < x \leqslant 2 \\ x - 3 & \text{if } x > 2 \end{cases}$$

(a) Evaluate each of the following, if it exists.

   (i) $\lim\limits_{x \to 1^-} g(x)$ (ii) $\lim\limits_{x \to 1} g(x)$ (iii) $g(1)$

   (iv) $\lim\limits_{x \to 2^-} g(x)$ (v) $\lim\limits_{x \to 2^+} g(x)$ (vi) $\lim\limits_{x \to 2} g(x)$

(b) Sketch the graph of g.

51. (a) If the symbol $[\![\,]\!]$ denotes the greatest integer function defined in Example 10, evaluate

   (i) $\lim\limits_{x \to -2^+} [\![x]\!]$ (ii) $\lim\limits_{x \to -2} [\![x]\!]$ (iii) $\lim\limits_{x \to -2.4} [\![x]\!]$

(b) If n is an integer, evaluate

   (i) $\lim\limits_{x \to n^-} [\![x]\!]$ (ii) $\lim\limits_{x \to n^+} [\![x]\!]$

(c) For what values of a does $\lim\limits_{x \to a} [\![x]\!]$ exist?

52. Let $f(x) = [\![\cos x]\!]$, $-\pi \leqslant x \leqslant \pi$.

(a) Sketch the graph of f.

(b) Evaluate each limit, if it exists.

   (i) $\lim\limits_{x \to 0} f(x)$ (ii) $\lim\limits_{x \to (\pi/2)^-} f(x)$

   (iii) $\lim\limits_{x \to (\pi/2)^+} f(x)$ (iv) $\lim\limits_{x \to \pi/2} f(x)$

(c) For what values of a does $\lim\limits_{x \to a} f(x)$ exist?

53. If $f(x) = [\![x]\!] + [\![-x]\!]$, show that $\lim\limits_{x \to 2} f(x)$ exists but is not equal to $f(2)$.

54. In the theory of relativity, the Lorentz contraction formula

$$L = L_0 \sqrt{1 - v^2/c^2}$$

expresses the length L of an object as a function of its velocity v with respect to an observer, where L_0 is the length of the object at rest and c is the speed of light. Find $\lim\limits_{v \to c^-} L$ and interpret the result. Why is a left-hand limit necessary?

55. If p is a polynomial, show that $\lim\limits_{x \to a} p(x) = p(a)$.

56. If r is a rational function, use Exercise 55 to show that $\lim\limits_{x \to a} r(x) = r(a)$ for every number a in the domain of r.

57. If $\lim\limits_{x \to 1} \dfrac{f(x) - 8}{x - 1} = 10$, find $\lim\limits_{x \to 1} f(x)$.

58. If $\lim\limits_{x \to 0} \dfrac{f(x)}{x^2} = 5$, find the following limits.

   (a) $\lim\limits_{x \to 0} f(x)$ (b) $\lim\limits_{x \to 0} \dfrac{f(x)}{x}$

59. If

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

prove that $\lim\limits_{x \to 0} f(x) = 0$.

60. Show by means of an example that $\lim\limits_{x \to a} [f(x) + g(x)]$ may exist even though neither $\lim\limits_{x \to a} f(x)$ nor $\lim\limits_{x \to a} g(x)$ exists.

61. Show by means of an example that $\lim\limits_{x \to a} [f(x) g(x)]$ may exist even though neither $\lim\limits_{x \to a} f(x)$ nor $\lim\limits_{x \to a} g(x)$ exists.

62. Evaluate $\lim\limits_{x \to 2} \dfrac{\sqrt{6 - x} - 2}{\sqrt{3 - x} - 1}$.

63. Is there a number a such that

$$\lim\limits_{x \to -2} \dfrac{3x^2 + ax + a + 3}{x^2 + x - 2}$$

exists? If so, find the value of a and the value of the limit.

64. The figure shows a fixed circle C_1 with equation $(x - 1)^2 + y^2 = 1$ and a shrinking circle C_2 with radius r and center the origin. P is the point $(0, r)$, Q is the upper point of intersection of the two circles, and R is the point of intersection of the line PQ and the x-axis. What happens to R as C_2 shrinks, that is, as $r \to 0^+$?