Review of Trigonometry for Calculus

Analytic Trigonometry

Basic Trigonometric Identities

FUNDAMENTAL TRIGONOMETRIC IDENTITIES

Reciprocal Identities

$$\csc x = \frac{1}{\sin x} \qquad \sec x = \frac{1}{\cos x} \qquad \cot x = \frac{1}{\tan x}$$

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1 \qquad \tan^2 x + 1 = \sec^2 x \qquad 1 + \cot^2 x = \csc^2 x$$

Even-Odd Identities

$$\sin(-x) = -\sin x \qquad \cos(-x) = \cos x \qquad \tan(-x) = -\tan x$$

Cofunction Identities

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x \qquad \tan\left(\frac{\pi}{2} - x\right) = \cot x \qquad \sec\left(\frac{\pi}{2} - x\right) = \csc x$$

$$\cos\left(\frac{\pi}{2} - x\right) = \sin x \qquad \cot\left(\frac{\pi}{2} - x\right) = \tan x \qquad \csc\left(\frac{\pi}{2} - x\right) = \sec x$$

Example 1 – *Simplifying a Trigonometric Expression*

Simplify the expression cos *t* + tan *t* sin *t*.

Solution:

We start by rewriting the expression in terms of sine and cosine.

$$\cos t + \tan t \sin t = \cos t + \left(\frac{\sin t}{\cos t}\right)\sin t \qquad \text{Reciprocal identity}$$

$$= \frac{\cos^2 t + \sin^2 t}{\cos t} \qquad \text{Common denominator}$$

$$= \frac{1}{\cos t} \qquad \text{Pythagorean identity}$$

$$= \sec t \qquad \text{Reciprocal identity}$$

Addition and Subtraction Formulas

We now derive identities for trigonometric functions of sums and differences.

ADDITION AND SUBTRACTION FORMULAS

Formulas for sine:

$$\sin(s + t) = \sin s \cos t + \cos s \sin t$$

$$\sin(s - t) = \sin s \cos t - \cos s \sin t$$

Formulas for cosine:

$$\cos(s + t) = \cos s \cos t - \sin s \sin t$$

$$\cos(s - t) = \cos s \cos t + \sin s \sin t$$

Formulas for tangent:

$$\tan(s + t) = \frac{\tan s + \tan t}{1 - \tan s \tan t}$$

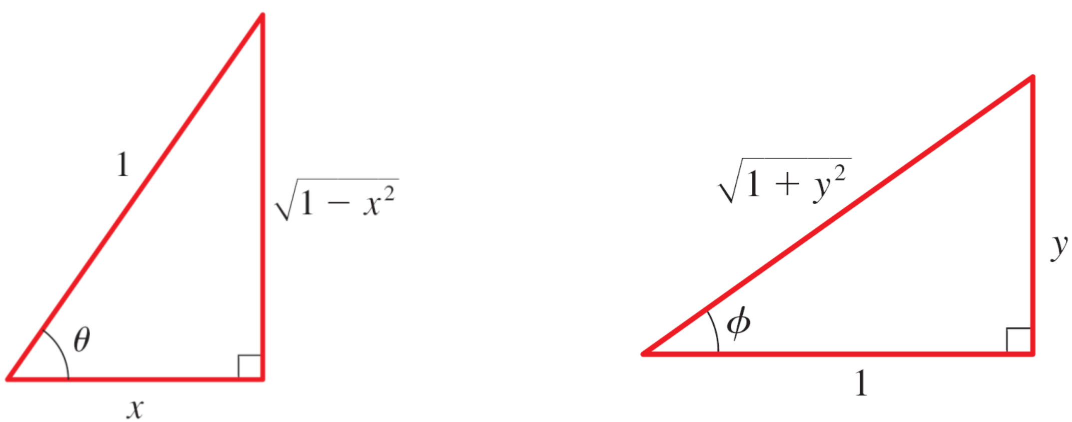
$$\tan(s - t) = \frac{\tan s - \tan t}{1 + \tan s \tan t}$$

Example 2– *Simplifying an Expression Involving Inverse Trigonometric Functions*

Write $\sin(\cos^{-1} x + \tan^{-1} y)$ as an algebraic expression in x and y, where $-1 \leq x \leq 1$ and y is any real number.

Solution:

Let $\theta = \cos^{-1} x$ and $\phi = \tan^{-1} y$. We sketch triangles with angles θ and ϕ such that $\cos\theta = x$ and $\tan \phi = y$ (see Figure 2).



$\cos \theta = x$ $\tan \phi = y$

Figure 2

Example 2 – *Solution*

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From the triangles we have

$$\sin \theta = \sqrt{1 - x^2} \qquad \cos \phi = \frac{1}{\sqrt{1 + y^2}} \qquad \sin \phi = \frac{y}{\sqrt{1 + y^2}}$$

From the Addition Formula for Sine we have

$$\sin(\cos^{-1} x + \tan^{-1} y) = \sin(\theta + \phi)$$

$$= \sin \theta \cos \phi + \cos \theta \sin \phi$$

Addition Formula for Sine

Example 2 – *Solution*

cont'd

$$= \sqrt{1 - x^2}\,\frac{1}{\sqrt{1 + y^2}} + x\,\frac{y}{\sqrt{1 + y^2}}$$

From triangles

$$= \frac{1}{\sqrt{1 + y^2}}\left(\sqrt{1 - x^2} + xy\right)$$

Factor $\dfrac{1}{\sqrt{1 + y^2}}$

Double-Angle Formulas

The formulas in the following box are immediate consequences of the addition formulas.

DOUBLE-ANGLE FORMULAS

Formula for sine: $\sin 2x = 2 \sin x \cos x$

Formulas for cosine:

$$\begin{aligned} \cos 2x &= \cos^2 x - \sin^2 x \\ &= 1 - 2\sin^2 x \\ &= 2\cos^2 x - 1 \end{aligned}$$

Formula for tangent: $\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$

Half-Angle Formulas

The following formulas allow us to write any trigonometric expression involving even powers of sine and cosine in terms of the first power of cosine only.

This technique is important in calculus. The Half-Angle Formulas are immediate consequences of these formulas.

FORMULAS FOR LOWERING POWERS

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\tan^2 x = \frac{1 - \cos 2x}{1 + \cos 2x}$$

Example 3 – *Lowering Powers in a Trigonometric Expression using Double Angle and Half Angle formulas*

Express $\sin^2 x \cos^2 x$ in terms of the first power of cosine.

Solution:

We use the formulas for lowering powers repeatedly.

$$\sin^2 x \cos^2 x = \left(\frac{1 - \cos 2x}{2}\right)\left(\frac{1 + \cos 2x}{2}\right)$$

$$= \frac{1 - \cos^2 2x}{4} = \frac{1}{4} - \frac{1}{4}\cos^2 2x$$

$$= \frac{1}{4} - \frac{1}{4}\left(\frac{1 + \cos 4x}{2}\right) = \frac{1}{4} - \frac{1}{8} - \frac{\cos 4x}{8}$$

$$= \frac{1}{8} - \frac{1}{8}\cos 4x = \frac{1}{8}(1 - \cos 4x)$$

Example 3 – *Solution*

cont'd

Another way to obtain this identity is to use the Double-Angle Formula for Sine in the form $\sin x \cos x = \frac{1}{2} \sin 2x$. Thus

$$\sin^2 x \cos^2 x = \frac{1}{4} \sin^2 2x = \frac{1}{4}\left(\frac{1 - \cos 4x}{2}\right)$$

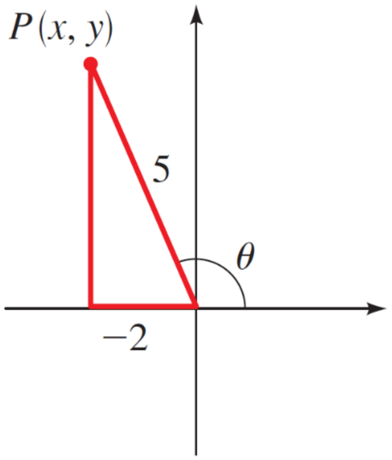
$$= \frac{1}{8}(1 - \cos 4x)$$

Example 4 – *Evaluating an Expression Involving Inverse Trigonometric Functions*

Evaluate sin 2θ, where cos $\theta = -\frac{2}{5}$ with θ in Quadrant II.

Solution :

We first sketch the angle θ in standard position with terminal side in Quadrant II:



Since cos $\theta = x/r = -\frac{2}{5}$,
we can label a side and the hypotenuse of the triangle

To find the remaining side, we use the Pythagorean Theorem.

Example 4 – *Solution*

cont'd

$x^2 + y^2 = r^2$	Pythagorean Theorem
$(-2)^2 + y^2 = 5^2$	$x = -2$, $r = 5$
$y = \pm\sqrt{21}$	Solve for y^2
$y = +\sqrt{21}$	Because $y > 0$

We can now use the Double-Angle Formula for Sine.

$\sin 2\theta = 2 \sin \theta \cos \theta$	Double-Angle Formula
$= 2\left(\frac{\sqrt{21}}{5}\right)\left(-\frac{2}{5}\right)$	From the triangle
$= -\frac{4\sqrt{21}}{25}$	Simplify

Basic Trigonometric Equations

Example 5: Basic Trigonometric Equations

Find all solutions of the equations:

(a) $2 \sin \theta - 1 = 0$ **(b)** $\tan^2 \theta - 3 = 0$

Solution:

(a) We start by isolating $\sin \theta$.

$$2 \sin \theta - 1 = 0 \qquad \text{Given equation}$$

$$2 \sin \theta = 1 \qquad \text{Add 1}$$

$$\sin \theta = \frac{1}{2} \qquad \text{Divide by 2}$$

Example 5 - Solution

The solutions are

$$\theta = \frac{\pi}{6} + 2k\pi \qquad \theta = \frac{5\pi}{6} + 2k\pi$$

where *k* is any integer.

(b) We start by isolating tan θ.

$$\tan^2 \theta - 3 = 0$$ Given equation

$$\tan^2 \theta = 3$$ Add 3

$$\tan \theta = \pm\sqrt{3}$$ Take the square root

Example 5 - Solution

cont'd

Because tangent has period π, we first find the solutions in any interval of length π. In the interval $(-\pi/2, \pi/2)$ the solutions are $\theta = \pi/3$ and $\theta = -\pi/3$.

To get all solutions, we add integer multiples of π to these solutions:

$$\theta = \frac{\pi}{3} + k\pi \qquad\qquad \theta = -\frac{\pi}{3} + k\pi$$

where k is any integer.

Solving Trigonometric Equations by Factoring

Example 6 – *A Trigonometric Equation of Quadratic Type*

Solve the equation $2\cos^2\theta - 7\cos\theta + 3 = 0$.

Solution:

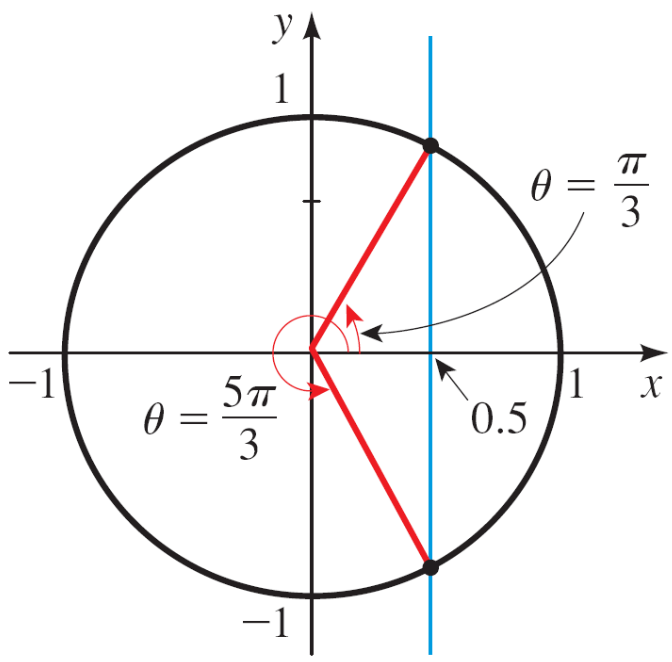
We factor the left-hand side of the equation.

$2\cos^2\theta - 7\cos\theta + 3 = 0$	Given equation
$(2\cos\theta - 1)(\cos\theta - 3) = 0$	Factor
$2\cos\theta - 1 = 0$ or $\cos\theta - 3 = 0$	Set each factor equal to 0
$\cos\theta = \frac{1}{2}$ or $\cos\theta = 3$	Solve for $\cos\theta$

Example 6 – *Solution*

cont'd

Because cosine has period 2π, we first find the solutions in the interval $[0, 2\pi)$. For the first equation the solutions are $\theta = \pi/3$ and $\theta = 5\pi/3$



Example 6 – *Solution*

cont’d

The second equation has no solution because cos θ is never greater than 1.

Thus the solutions are

$$\theta = \frac{\pi}{3} + 2k\pi \qquad \theta = \frac{5\pi}{3} + 2k\pi$$

where *k* is any integer.

Example 7 – *Solving a Trigonometric Equation by Factoring*

Solve the equation $5 \sin \theta \cos \theta + 4 \cos \theta = 0$.

Solution:

We factor the left-hand side of the equation.

$5 \sin \theta \cos \theta + 2 \cos \theta = 0$ Given equation

$\cos \theta (5 \sin \theta + 2) = 0$ Factor

$\cos \theta = 0$ or $5 \sin \theta + 4 = 0$ Set each factor equal to 0

$\sin \theta = -0.8$ Solve for $\sin \theta$

Example 7 – *Solution*

cont'd

Because sine and cosine have period 2π, we first find the solutions of these equations in an interval of length 2π.

For the first equation the solutions in the interval $[0, 2\pi)$ are $\theta = \pi/2$ and $\theta = 3\pi/2$. To solve the second equation, we take $\sin^{-1}$ of each side.

$$\sin \theta = -0.80 \qquad \text{Second equation}$$

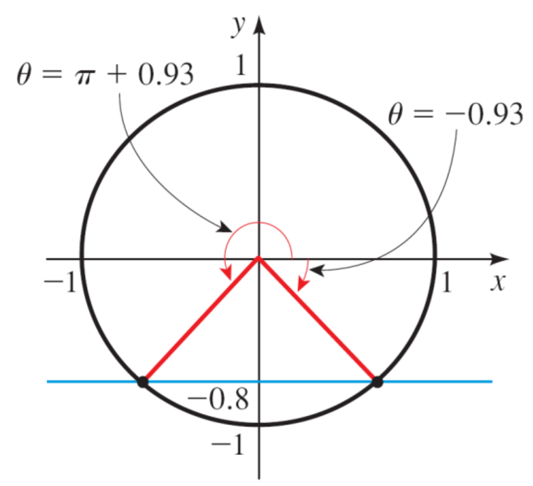
$$\theta = \sin^{-1}(-0.80) \qquad \text{Take } \sin^{-1} \text{ of each side}$$

Example 7 – *Solution*

cont'd

$\theta \approx -0.93$ Calculator (in radian mode)

So the solutions in an interval of length 2π are $\theta = -0.93$ and $\theta = \pi + 0.93 \approx 4.07$



Example 7 – *Solution*

cont'd

We get all the solutions of the equation by adding integer multiples of 2π to these solutions.

$$\theta = \frac{\pi}{2} + 2k\pi \qquad\qquad \theta = \frac{3\pi}{2} + 2k\pi$$

$$\theta \approx -0.93 + 2k\pi \qquad\qquad \theta \approx 4.07 + 2k\pi$$

where *k* is any integer.

Solving Trigonometric Equations by Using Identities

Example 8 – Solving by *Using a Trigonometric Identity*

Solve the equation $1 + \sin\theta = 2\cos^2\theta$.

Solution:

We first need to rewrite this equation so that it contains only one trigonometric function. To do this, we use a trigonometric identity:

$$1 + \sin\theta = 2\cos^2\theta \qquad \text{Given equation}$$

$$1 + \sin\theta = 2(1 - \sin^2\theta) \qquad \text{Pythagorean identity}$$

$$2\sin^2\theta + \sin\theta - 1 = 0 \qquad \text{Put all terms on one side}$$

$$(2\sin\theta - 1)(\sin\theta + 1) = 0 \qquad \text{Factor}$$

Example 8 – *Solution*

cont'd

$2\sin\theta - 1 = 0$ or $\sin\theta + 1 = 0$ Set each factor equal to 0

$\sin\theta = \frac{1}{2}$ or $\sin\theta = -1$ Solve for $\sin\theta$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$ or $\theta = \frac{3\pi}{2}$ Solve for θ in the interval $[0, 2\pi)$

Because sine has period 2π, we get all the solutions of the equation by adding integer multiples of 2π to these solutions.

Example 8 – *Solution*

cont'd

Thus the solutions are

$$\theta = \frac{\pi}{6} + 2k\pi \qquad \theta = \frac{5\pi}{6} + 2k\pi \qquad \theta = \frac{3\pi}{2} + 2k\pi$$

where *k* is any integer.

Solving Trigonometric Equations by Graphs on calculator

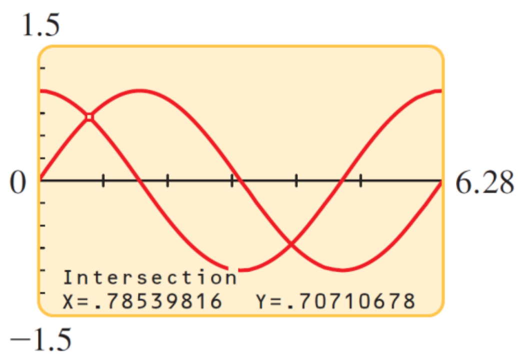
Example 9 – Solving Graphically by *Finding Intersection Points*

Find the values of x for which the graphs of $f(x) = \sin x$ and $g(x) = \cos x$ intersect.

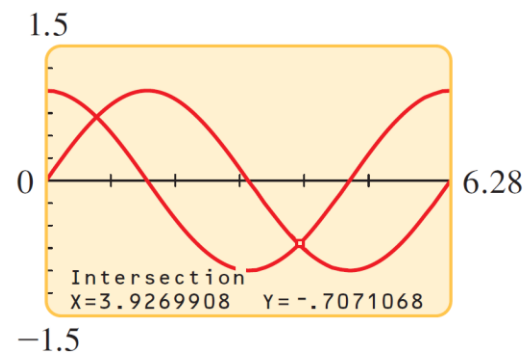
Solution :

The graphs intersect where $f(x) = g(x)$.

We graph $y_1 = \sin x$ and $y_2 = \cos x$ on the same screen, for x between 0 and 2π.



(a)



(b)

Example 9 – *Solution*

cont'd

Using $\boxed{\text{TRACE}}$ or the intersect command on the graphing calculator, we see that the two points of intersection in this interval occur where $x \approx 0.785$ and $x \approx 3.927$.

Since sine and cosine are periodic with period 2π, the intersection points occur where

$$x \approx 0.785 + 2k\pi \qquad \text{and} \qquad x \approx 3.927 + 2k\pi$$

where k is any integer.

Equations with Trigonometric Functions of Multiples of Angles

Example 10 – *A Trigonometric Equation Involving a Multiple of an Angle*

Consider the equation $2 \sin 3\theta - 1 = 0$.

(a) Find all solutions of the equation.

(b) Find the solutions in the interval $[0, 2\pi)$.

Solution:

(a) We first isolate $\sin 3\theta$ and then solve for the angle 3θ.

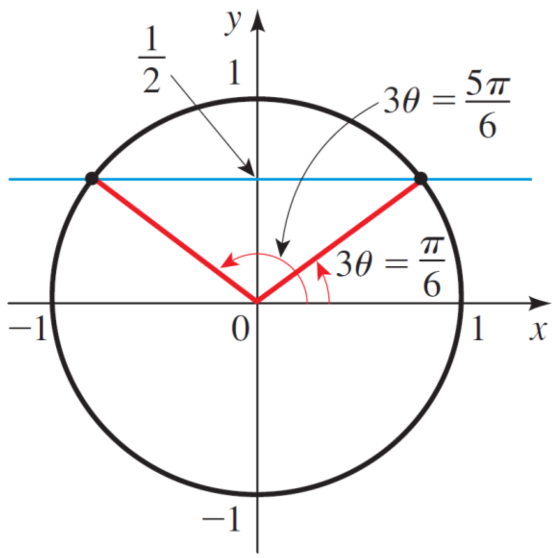
$2 \sin 3\theta - 1 = 0$	Given equation
$2 \sin 3\theta = 1$	Add 1
$\sin 3\theta = \frac{1}{2}$	Divide by 2

Example 10 – *Solution*

cont'd

$$3\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

Solve for 3θ in the interval $[0, 2\pi)$



Example 10 – *Solution*

cont'd

To get all solutions, we add integer multiples of 2π to these solutions. So the solutions are of the form

$$3\theta = \frac{\pi}{6} + 2k\pi \qquad 3\theta = \frac{5\pi}{6} + 2k\pi$$

To solve for θ, we divide by 3 to get the solutions

$$\theta = \frac{\pi}{18} + \frac{2k\pi}{3} \qquad \theta = \frac{5\pi}{18} + \frac{2k\pi}{3}$$

where k is any integer.

Example 10 – *Solution*

cont'd

(b) The solutions from part (a) that are in the interval $[0, 2\pi)$ correspond to $k = 0, 1$, and 2. For all other values of k the corresponding values of θ lie outside this interval.

So the solutions in the interval $[0, 2\pi)$ are

$$\theta = \underbrace{\frac{\pi}{18}, \frac{5\pi}{18}}_{k=0}, \underbrace{\frac{13\pi}{18}, \frac{17\pi}{18}}_{k=1}, \underbrace{\frac{25\pi}{18}, \frac{29\pi}{18}}_{k=2}$$