SEQUENCES: $\{a_n\}$

A sequence $\{a_n\}$ is said to **converge** to a number L if $\lim_{n\to\infty} a_n = L$. If $\lim_{n\to\infty} a_n = \infty$, $\lim_{n\to\infty} a_n = -\infty$, or Does Not Exist (DNE), then the sequence **diverges**.

SERIES: $\sum_{k=1}^{\infty} a_k$

The series, $\sum_{k=1}^{\infty} a_k$, is said to **converge** if there is a number S such that $\sum_{k=1}^{\infty} a_k = S$. Otherwise the series **diverges**. There are several tests you can use to determine whether or not a series converges or diverges. Some tests that prove a series converges will not give you the particular S that the series converges to, but that is OK. At least you know that the series converges.

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The first thing that you should do is look at $\lim_{k\to\infty} a_k$. If $\lim_{k\to\infty} a_k \neq 0$, then the series **diverges** by the **nth Term Test For Divergence**. If $\lim_{k\to\infty} a_k = 0$, then this does not tell you anything about convergence or divergence, and you must use another test to determine convergence or divergence. Look at the harmonic series, described later, to see a series that diverges, but has $\lim_{k\to\infty} a_k = 0$.

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If you have a "collapsing" series, like $\sum_{k=1}^{\infty}\left[\frac{k}{k+2} - \frac{k+1}{k+3}\right]$ for example, then you will look at S_n, the **nth Partial Sum** of the series. Since $S_n = \sum_{k=1}^{n} a_k$, you get that $\lim_{n\to\infty} S_n = \lim_{n\to\infty}\sum_{k=1}^{n} a_k = \sum_{k=1}^{\infty} a_k$. Therefore, if the $\lim_{n\to\infty} S_n = S$ for some finite number S, then the series **converges** to S, i.e. $\sum_{k=1}^{\infty} a_k = S$. If $\lim_{n\to\infty} S_n = \infty, -\infty$ or DNE, then the series **diverges**.

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GEOMETRIC SERIES: Geometric series are all of the form $a\sum_{k=1}^{\infty} r^{k-1} = a + ar + ar^2 + ar^3 + \cdots$, where a is a nonzero constant and r is another constant called the *ratio*.

$S_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1}$ $(r \neq 1)$ is the nth partial sum of the geometric series. A geometric series **converges** to $S = \frac{a}{1-r}$, if $|r| < 1$. Otherwise, the geometric series **diverges**.

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The following tests are for positive series (terms are all nonnegative).

BOUNDED SUM TEST: Let $\sum_{k=1}^{\infty} a_k$ be a series with nonnegative terms. If the sequence $\{S_n\}$ of nth partial sums is unbounded, then the series **diverges**. If there is a constant M, such that $S_n \leq M$ for all n, then the series converges and has sum $S \leq M$.

COMPARISON TEST: The goal here is to try to compare the unknown series to a series that you already know converges or diverges or to a series on which you can easily use another test to determine its convergence or divergence. Suppose $0 \le a_k \le b_k$ for $k = 1,2,3,4,\ldots$

(i) If $\sum_{k=1}^{\infty} b_k$ **converges**, then $\sum_{k=1}^{\infty} a_k$ **converges** also.

(ii) If $\sum_{k=1}^{\infty} a_k$ **diverges**, then $\sum_{k=1}^{\infty} b_k$ **diverges** also.

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RATIO TEST: This test is used when the terms in the series involve factorials and/or powers of one or more constants (b^k, $k!$ for example). **Note:** The ratio test is also used to find the interval of convergence for **power series** $\sum_{k=1}^{\infty} c_k x^k$. Let $\sum_{k=1}^{\infty} a_k$ be a series of **positive** terms and suppose that

$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = r$, (i.e. the limit exists). For power series it is necessary to take $\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|}$.

(i) If $r < 1$, the series **converges.**
(ii) If $r > 1$, the series **diverges.**
(iii) If $r = 1$, the test is inconclusive.

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INTEGRAL TEST: The integral test should be your last resort when trying to decide whether a given positive series is convergent or divergent. Consider the positive series $\sum_{k=b}^{\infty} a_k$. Let $f(x)$ be a continuous, positive, nonincreasing function on the interval $[b,\infty)$ and suppose $a_k = f(k)$ for all integers $k \ge b$. Then the series $\sum_{k=b}^{\infty} a_k$ and the integral $\int_b^{\infty} f(x)\,dx$ converge or diverge **together.**

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Using the integral test, it can be shown that a **p-series,** $\sum_{k=1}^{\infty} \frac{1}{k^p}$, **converges** for $p > 1$ and **diverges** for $p \le 1$. Notice that for $p = 1$, the result is the famous **harmonic series** $\sum_{k=1}^{\infty} \frac{1}{k}$, which diverges.

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The series are no longer assumed to be positive.

ABSOLUTE CONVERGENCE TEST: If a series **converges absolutely,** then it converges. That is, if $\sum_{k=1}^{\infty} |a_k|$ converges, then $\sum_{k=1}^{\infty} a_k$ converges (absolutely).

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ALTERNATING SERIES TEST: Let $b_1 - b_2 + b_3 - b_4 + b_5 - \cdots = \sum_{k=1}^{\infty} (-1)^{k-1} b_k$ be an alternating series with $b_k > b_{k+1} > 0$. If $\lim_{k\to\infty} b_k = 0$, then the series **converges.**

If $\sum_{k=1}^{\infty} a_k$ converges, but $\sum_{k=1}^{\infty} |a_k|$ does not converge, then $\sum_{k=1}^{\infty} a_k$ is said to **converge conditionally.**