Review

Algebra

Section R.1

Polynomials

Properties of Real Numbers

For all real numbers a, b, and c:

1. $a + b = b + a$; $ab = ba$; — **Commutative properties**
2. $(a + b) + c = a + (b + c)$; $(ab)c = a(bc)$; — **Associative properties**
3. $a(b + c) = ab + ac$. — **Distributive property**

Your Turn 1

Perform the operation $3(x^2 - 4x - 5) - 4(3x^2 - 5x - 7)$.

Solution: Multiply each polynomial by the coefficient in front of the polynomial and then combine like terms.

$$= 3x^2 - 12x - 15 - 12x^2 + 20x + 28$$

$$= -9x^2 + 8x + 13$$

Your Turn 2

Perform the operation $(3y+2)(4y^2-2y-5)$.

Solution : Using the Distributive property yields

$$=3y(4y^2-2y-5)+2(4y^2-2y-5)$$

$$=3y(4y^2)+3y(-2y)+3y(-5)+2(4y^2)+2(-2y)+2(-5)$$

$$=12y^3-6y^2-15y+8y^2-4y-10$$

$$=12y^3+2y^2-19y-10.$$

Section R.2

Factoring

Your Turn 1

Factor out the greatest common factor in $4z^4 + 4z^3 + 18z^2$.

Solution: Each of these terms is divisible by $2z^2$.

$$4z^4 + 4z^3 + 18z^2 = 2z^2(2z^2) + 2z^2(2z) + 2z^2(9)$$

$$= 2z^2(2z^2 + 2z + 9).$$

Special Factorizations

$x^2 - y^2 = (x + y)(x - y)$	**Difference of two squares**
$x^2 + 2xy + y^2 = (x + y)^2$	**Perfect square**
$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$	**Difference of two cubes**
$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$	**Sum of two cubes**

Example 4

Factor each polynomial, if possible.

(a) $64p^2 - 49q^2 = (8p)^2 - (7q)^2 = (8p + 7q)(8p - 7q)$ Difference of two squares

(b) $x^2 + 36$ is a prime polynomial.

(c) $x^2 + 12x + 36 = (x + 6)^2$ Perfect square

(d) $9y^2 - 24yz + 16z^2 = (3y - 4z)^2$ Perfect square

(e) $y^3 - 8 = y^3 - 2^3 = (y - 2)(y^2 + 2y + 4)$ Difference of two cubes

(f) $m^3 + 125 = m^3 + 5^3 = (m + 5)(m^2 - 5m + 25)$ Sum of two cubes

(g) $8k^3 - 27z^3 = (2k)^3 - (3z)^3 = (2k - 3z)(4k^2 + 6kz + 9z^2)$ Difference of two cubes

(h) $p^4 - 1 = (p^2 + 1)(p^2 - 1) = (p^2 + 1)(p + 1)(p - 1)$ Difference of two squares

Section R.3

Rational Expressions

Properties of Rational Expressions

For all mathematical expressions P, Q, R, and S, with $Q \neq 0$ and $S \neq 0$:

$\frac{P}{Q} = \frac{PS}{QS}$	**Fundamental property**
$\frac{P}{Q} + \frac{R}{Q} = \frac{P + R}{Q}$	**Addition**
$\frac{P}{Q} - \frac{R}{Q} = \frac{P - R}{Q}$	**Subtraction**
$\frac{P}{Q} \cdot \frac{R}{S} = \frac{PR}{QS}$	**Multiplication**
$\frac{P}{Q} \div \frac{R}{S} = \frac{P}{Q} \cdot \frac{S}{R} \quad (R \neq 0)$	**Division**

Your Turn 1

Write in lowest terms $\dfrac{z^2+5z+6}{2z^2+7z+3}$.

Solution: Factor both numerator and denominator in order to identify any common factors.

$$=\frac{(z+2)(z+3)}{(2z+1)(z+3)}$$

$$=\frac{(z+2)}{(2z+1)}.$$

The answer can not be further simplified.

Your Turn 2

Perform the following operations $\dfrac{z^2+5z+6}{2z^2-5z-3}\cdot\dfrac{2z^2-z-1}{z^2+2z-3}$.

Solution: Factor where possible.

$$=\frac{(z+2)(z+3)}{(2z+1)(z-3)}\cdot\frac{(2z+1)(z-1)}{(z+3)(z-1)}$$

$$=\frac{(z+2)}{(z-3)}$$

Section R.4

Equations

Properties of Equality

For all real numbers a, b, and c:

1. **If $a = b$, then $a + c = b + c$.** — **Addition property of equality**
   (The same number may be added to both sides of an equation.)
2. **If $a = b$, then $ac = bc$.** — **Multiplication property of equality**
   (Both sides of an equation may be multiplied by the same number.)

Your Turn 1

Solve $3x - 7 = 4(5x + 2) - 7x$.

Solution: $3x - 7 = 20x + 8 - 7x$ Distributive Property

$3x - 7 = 13x + 8$ Combine the like terms.

$-10x - 7 = 8$ Add $-13x$ to both sides.

$-10x = 15$ Add 7 to both sides.

$\frac{-10x}{-10} = \frac{15}{-10}$ Multiply both sides by $\frac{1}{-10}$.

$x = -\frac{3}{2}$

Zero-Factor Property

If a and b are real numbers, with $ab = 0$, then either

$$a = 0 \quad \text{or} \quad b = 0 \qquad \text{(or both).}$$

Your Turn 2

Solve $2m^2 + 7m = 15.$

Solution: First write the equation in standard form.

$$2m^2 + 7m - 15 = 0$$

Now factor $2m^2 + 7m - 15$ to get

$$(2m - 3)(m + 5) = 0.$$

By the zero-factor property, the product $(2m - 3)(m + 5)$ can equal 0 if and only if

$2m - 3 = 0$ or $m + 5 = 0.$

Solve each of these equations separately to find that the solutions are $\frac{3}{2}$ and $-5.$

Quadratic Formula

The solutions of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Your Turn 3

Solve $z^2 + 6 = 8z$.

Solution: First, add $-8z$ on both sides of the equal sign in order to get the equation in standard form.

$z^2 - 8z + 6 = 0.$

Now identify the letters a, b, and c.

Here, $a = 1$, $b = -8$, and $c = 6$.

Substitute these numbers into the quadratic formula.

$$x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(6)}}{2(1)}$$

Continued

Your Turn 3 continued

$$x = \frac{8 \pm \sqrt{64 - 24}}{2}$$

$$x = \frac{8 \pm \sqrt{40}}{2} = \frac{8 \pm 2\sqrt{10}}{2}$$

$$= \frac{2(4 \pm \sqrt{10})}{2}$$ Factor $8 \pm 2\sqrt{10}$

$$= 4 \pm \sqrt{10}$$ Reduce to lowest terms.

The two solutions are $4+\sqrt{10}$ and $4-\sqrt{10}$.

Your Turn 4

Solve $\frac{1}{x^2-4}+\frac{2}{x-2}=\frac{1}{x}.$

Solution: Factor x^2-4 as $(x-2)(x+2).$

$$\frac{1}{(x-2)(x+2)}+\frac{2}{x-2}=\frac{1}{x}$$

The least common denominator for all the fractions is $x(x+2)(x-2)$. Multiplying both sides by $x(x+2)(x-2)$ gives the following:

$$x(x-2)(x+2)\left(\frac{1}{(x-2)(x+2)}+\frac{2}{x-2}\right)=x(x-2)(x+2)\frac{1}{x}$$

Continued

Your Turn 4 continued

$$x + 2x(x+2) = (x-2)(x+2)$$

$$x + 2x^2 + 4x = x^2 + 2x - 2x - 4$$ Distributive property

$$2x^2 + 5x = x^2 - 4$$

Add $-x^2$ and 4, Rearrange terms.

$$x^2 + 5x + 4 = 0$$

$$(x+1)(x+4) = 0$$

$x + 1 = 0$ or $x + 4 = 0$

$x = -1$ $\quad$ $x = -4$

Verify that the solutions are -1 and -4.

Section R.5

Inequalities

Inequality Symbols

$<$ means *is less than*

$>$ means *is greater than*

$\leq$ means *is less than or equal to*

$\geq$ means *is greater than or equal to*

Properties of Inequality

For all real numbers a, b, and c:

1. If $a < b$, then $a + c < b + c$.
2. If $a < b$ and if $c > 0$, then $ac < bc$.
3. If $a < b$ and if $c < 0$, then $ac > bc$.

Your Turn 1

Solve $3z - 2 > 5z + 7$.

Solution: $3z - 2 + 2 > 5z + 7 + 2$ Add 2 to both sides.

$$3z > 5z + 9$$

$$3z + (-5z) > 5z + 9 + (-5z)$$ Add $-5z$ to both sides.

$$-2z > 9$$

$$\frac{1}{-2} \cdot -2z < \frac{1}{-2} \cdot 9$$

Multiplying by a negative changes the direction of the inequality.

$$z < -\frac{9}{2}$$

Section R.6

Exponents

Definition of Exponent

If n is a natural number, then

$$a^n = a \cdot a \cdot a \cdot \cdots \cdot a,$$

where a appears as a factor n times.

Zero and Negative Exponents

If a is any nonzero real number, and if n is a positive integer, then

$$a^0 = 1 \qquad \text{and} \qquad a^{-n} = \frac{1}{a^n}.$$

Properties of Exponents

For any integers m and n, and any real numbers a and b for which the following exist:

1. $a^m \cdot a^n = a^{m+n}$
2. $\dfrac{a^m}{a^n} = a^{m-n}$
3. $(a^m)^n = a^{mn}$
4. $(ab)^m = a^m \cdot b^m$
5. $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$

Your Turn

Simplify $$\left(\frac{y^{2}z^{-4}}{y^{-3}z^{4}}\right)^{-2}$$

Solution: $$=\frac{(y^{2})^{-2}(z^{-4})^{-2}}{(y^{-3})^{-2}(z^{4})^{-2}}$$ Property 4 and 5

$$=\frac{y^{-4}z^{8}}{y^{6}z^{-8}}$$ Property 3

$$=\frac{z^{8-(-8)}}{y^{6-(-4)}}$$ Property 2

$$=\frac{z^{16}}{y^{10}}.$$

Definition of $a^{m/n}$

For all real numbers a for which the indicated roots exist, and for any rational number m/n,

$$a^{m/n} = (a^{1/n})^m.$$

Section R.7

Radicals

Radicals

If n is an even natural number and $a > 0$, or n is an odd natural number, then

$$\boldsymbol{a^{1/n} = \sqrt[n]{a}}.$$

Properties of Radicals

For all real numbers a and b and natural numbers m and n such that $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers:

1. $(\sqrt[n]{a})^n = a$

2. $\sqrt[n]{a^n} = \begin{cases} |a| & \text{if } n \text{ is even} \\ a & \text{if } n \text{ is odd} \end{cases}$

3. $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$

4. $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}} \quad (b \neq 0)$

5. $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$

Your Turn

Rationalize the denominator in $\dfrac{5}{\sqrt{x}-\sqrt{y}}$.

Solution: The best approach here is to multiply both numerator and denominator by $\sqrt{x} + \sqrt{y}$. The expressions $\sqrt{x} + \sqrt{y}$ and $\sqrt{x} - \sqrt{y}$ are conjugates, and their product is

$$(\sqrt{x})^2 - (\sqrt{y})^2 = x - y.$$

Thus,

$$= \frac{5(\sqrt{x}+\sqrt{y})}{(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})} = \frac{5\sqrt{x}+5\sqrt{y}}{x-y}.$$