6.8 Exercises

1–4 Given that

$$\lim_{x \to a} f(x) = 0 \qquad \lim_{x \to a} g(x) = 0 \qquad \lim_{x \to a} h(x) = 1$$

$$\lim_{x \to a} p(x) = \infty \qquad \lim_{x \to a} q(x) = \infty$$

which of the following limits are indeterminate forms? For those that are not an indeterminate form, evaluate the limit where possible.

1. (a) $\lim_{x \to a} \frac{f(x)}{g(x)}$ (b) $\lim_{x \to a} \frac{f(x)}{p(x)}$ (c) $\lim_{x \to a} \frac{h(x)}{p(x)}$

(d) $\lim_{x \to a} \frac{p(x)}{f(x)}$ (e) $\lim_{x \to a} \frac{p(x)}{q(x)}$

2. (a) $\lim_{x \to a} [f(x)p(x)]$ (b) $\lim_{x \to a} [h(x)p(x)]$

(c) $\lim_{x \to a} [p(x)q(x)]$

3. (a) $\lim_{x \to a} [f(x) - p(x)]$ (b) $\lim_{x \to a} [p(x) - q(x)]$

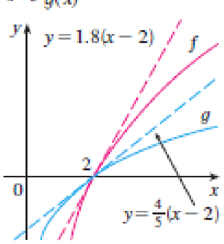
(c) $\lim_{x \to a} [p(x) + q(x)]$

4. (a) $\lim_{x \to a} [f(x)]^{g(x)}$ (b) $\lim_{x \to a} [f(x)]^{p(x)}$ (c) $\lim_{x \to a} [h(x)]^{p(x)}$

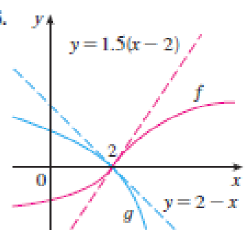
(d) $\lim_{x \to a} [p(x)]^{f(x)}$ (e) $\lim_{x \to a} [p(x)]^{q(x)}$ (f) $\lim_{x \to a} \sqrt[q(x)]{p(x)}$

5–6 Use the graphs of f and g and their tangent lines at (2, 0) to find $\lim_{x \to 2} \frac{f(x)}{g(x)}$.

5.



6.



7–66 Find the limit. Use l'Hospital's Rule where appropriate. If there is a more elementary method, consider using it. If l'Hospital's Rule doesn't apply, explain why.

7. $\lim_{x \to 1} \frac{x^2 - 1}{x^2 - x}$

8. $\lim_{x \to 2} \frac{x^2 + x - 6}{x - 2}$

9. $\lim_{x \to 1} \frac{x^3 - 2x^2 + 1}{x^3 - 1}$

10. $\lim_{x \to 1/2} \frac{6x^2 + 5x - 4}{4x^2 + 16x - 9}$

11. $\lim_{x \to (\pi/2)^+} \frac{\cos x}{1 - \sin x}$

12. $\lim_{x \to 0} \frac{\sin 4x}{\tan 5x}$

13. $\lim_{t \to 0} \frac{e^{2t} - 1}{\sin t}$

14. $\lim_{x \to 0} \frac{x^2}{1 - \cos x}$

15. $\lim_{\theta \to \pi/2} \frac{1 - \sin \theta}{1 + \cos 2\theta}$

16. $\lim_{\theta \to \pi/2} \frac{1 - \sin \theta}{\csc \theta}$

17. $\lim_{x \to \infty} \frac{\ln x}{\sqrt{x}}$

18. $\lim_{x \to \infty} \frac{x + x^2}{1 - 2x^2}$

19. $\lim_{x \to 0^+} \frac{\ln x}{x}$

20. $\lim_{x \to \infty} \frac{\ln \sqrt{x}}{x^2}$

21. $\lim_{t \to 1} \frac{t^8 - 1}{t^5 - 1}$

22. $\lim_{t \to 0} \frac{8^t - 5^t}{t}$

23. $\lim_{x \to 0} \frac{\sqrt{1 + 2x} - \sqrt{1 - 4x}}{x}$

24. $\lim_{u \to \infty} \frac{e^{u/10}}{u^3}$

25. $\lim_{x \to 0} \frac{e^x - 1 - x}{x^2}$

26. $\lim_{x \to 0} \frac{\sinh x - x}{x^3}$

27. $\lim_{x \to 0} \frac{\tanh x}{\tan x}$

28. $\lim_{x \to 0} \frac{x - \sin x}{x - \tan x}$

29. $\lim_{x \to 0} \frac{\sin^{-1} x}{x}$

30. $\lim_{x \to \infty} \frac{(\ln x)^2}{x}$

31. $\lim_{x \to 0} \frac{x 3^x}{3^x - 1}$

32. $\lim_{x \to 0} \frac{\cos mx - \cos nx}{x^2}$

33. $\lim_{x \to 0} \frac{x + \sin x}{x + \cos x}$

34. $\lim_{x \to 0} \frac{x}{\tan^{-1}(4x)}$

35. $\lim_{x \to 1} \dfrac{1 - x + \ln x}{1 + \cos \pi x}$

36. $\lim_{x \to 0^+} \dfrac{x^x - 1}{\ln x + x - 1}$

37. $\lim_{x \to 1} \dfrac{x^a - ax + a - 1}{(x - 1)^2}$

38. $\lim_{x \to 0} \dfrac{e^x - e^{-x} - 2x}{x - \sin x}$

39. $\lim_{x \to 0} \dfrac{\cos x - 1 + \frac{1}{2}x^2}{x^4}$

40. $\lim_{x \to a^+} \dfrac{\cos x \ln(x - a)}{\ln(e^x - e^a)}$

41. $\lim_{x \to \infty} x \sin(\pi/x)$

42. $\lim_{x \to \infty} \sqrt{x}\, e^{-x/2}$

43. $\lim_{x \to 0} \cot 2x \sin 6x$

44. $\lim_{x \to 0^+} \sin x \ln x$

45. $\lim_{x \to \infty} x^3 e^{-x^2}$

46. $\lim_{x \to \infty} x \tan(1/x)$

47. $\lim_{x \to 1^+} \ln x \tan(\pi x/2)$

48. $\lim_{x \to (\pi/2)^-} \cos x \sec 5x$

49. $\lim_{x \to 1} \left(\dfrac{x}{x - 1} - \dfrac{1}{\ln x} \right)$

50. $\lim_{x \to 0} (\csc x - \cot x)$

51. $\lim_{x \to 0^+} \left(\dfrac{1}{x} - \dfrac{1}{e^x - 1} \right)$

52. $\lim_{x \to 0} \left(\cot x - \dfrac{1}{x} \right)$

53. $\lim_{x \to \infty} (x - \ln x)$

54. $\lim_{x \to 1^+} [\ln(x^7 - 1) - \ln(x^5 - 1)]$

55. $\lim_{x \to 0^+} x^{\sqrt{x}}$

56. $\lim_{x \to 0^+} (\tan 2x)^x$

57. $\lim_{x \to 0} (1 - 2x)^{1/x}$

58. $\lim_{x \to \infty} \left(1 + \dfrac{a}{x} \right)^{bx}$

59. $\lim_{x \to 1^+} x^{1/(1-x)}$

60. $\lim_{x \to \infty} x^{(\ln 2)/(1 + \ln x)}$

61. $\lim_{x \to \infty} x^{1/x}$

62. $\lim_{x \to \infty} (e^x + x)^{1/x}$

63. $\lim_{x \to 0^+} (4x + 1)^{\cot x}$

64. $\lim_{x \to 1} (2 - x)^{\tan(\pi x/2)}$

65. $\lim_{x \to 0^+} (\cos x)^{1/x^2}$

66. $\lim_{x \to \infty} \left(\dfrac{2x - 3}{2x + 5} \right)^{2x+1}$

67–68 Use a graph to estimate the value of the limit. Then use l'Hospital's Rule to find the exact value.

67. $\lim_{x \to \infty} \left(1 + \dfrac{2}{x} \right)^x$

68. $\lim_{x \to 0} \dfrac{5^x - 4^x}{3^x - 2^x}$

69–70 Illustrate l'Hospital's Rule by graphing both $f(x)/g(x)$ and $f'(x)/g'(x)$ near $x = 0$ to see that these ratios have the same limit as $x \to 0$. Also, calculate the exact value of the limit.

69. $f(x) = e^x - 1, \quad g(x) = x^3 + 4x$

70. $f(x) = 2x \sin x, \quad g(x) = \sec x - 1$

71. Prove that

$$\lim_{x \to \infty} \frac{e^x}{x^n} = \infty$$

for any positive integer n. This shows that the exponential function approaches infinity faster than any power of x.

72. Prove that

$$\lim_{x \to \infty} \frac{\ln x}{x^p} = 0$$

for any number $p > 0$. This shows that the logarithmic function approaches ∞ more slowly than any power of x.

73–74 What happens if you try to use l'Hospital's Rule to find the limit? Evaluate the limit using another method.

73. $\lim_{x \to \infty} \dfrac{x}{\sqrt{x^2 + 1}}$

74. $\lim_{x \to (\pi/2)^-} \dfrac{\sec x}{\tan x}$

75–80 Use l'Hospital's Rule to help sketch the curve. Use the guidelines of Section 3.5.

75. $y = xe^{-x}$

76. $y = \dfrac{\ln x}{x^2}$

77. $y = xe^{-x^2}$

78. $y = e^x/x$

79. $y = x - \ln(1 + x)$

80. $y = (x^2 - 3)e^{-x}$

CAS **81–83**

(a) Graph the function.

(b) Use l'Hospital's Rule to explain the behavior as $x \to 0^+$ or as $x \to \infty$.

(c) Estimate the maximum and minimum values and then use calculus to find the exact values.

(d) Use a graph of f'' to estimate the x-coordinates of the inflection points.

81. $f(x) = x^{-x}$

82. $f(x) = (\sin x)^{\sin x}$

83. $f(x) = x^{1/x}$

84. Investigate the family of curves given by $f(x) = x^n e^{-x}$, where n is a positive integer. What features do these curves have in common? How do they differ from one another? In particular, what happens to the maximum and minimum points and inflection points as n increases? Illustrate by graphing several members of the family.

85. Investigate the family of curves $f(x) = e^x - cx$. In particular, find the limits as $x \to \pm\infty$ and determine the values of c for which f has an absolute minimum. What happens to the minimum points as c increases?

86. If an object with mass m is dropped from rest, one model for its speed v after t seconds, taking air resistance into account, is

$$v = \frac{mg}{c}(1 - e^{-ct/m})$$

where g is the acceleration due to gravity and c is a positive constant. (In Chapter 9 we will be able to deduce this equation from the assumption that the air resistance is proportional to the speed of the object; c is the proportionality constant.)

(a) Calculate $\lim_{t \to \infty} v$. What is the meaning of this limit?

(b) For fixed t, use l'Hospital's Rule to calculate $\lim_{c \to 0^+} v$. What can you conclude about the velocity of a falling object in a vacuum?

87. If an initial amount A_0 of money is invested at an interest rate r compounded n times a year, the value of the investment after t years is

$$A = A_0\left(1 + \frac{r}{n}\right)^{nt}$$

If we let $n \to \infty$, we refer to the *continuous compounding* of interest. Use l'Hospital's Rule to show that if interest is compounded continuously, then the amount after t years is

$$A = A_0 e^{rt}$$

88. If a metal ball with mass m is projected in water and the force of resistance is proportional to the square of the velocity, then the distance the ball travels in time t is

$$s(t) = \frac{m}{c} \ln \cosh \sqrt{\frac{gc}{mt}}$$

where c is a positive constant. Find $\lim_{c \to 0^+} s(t)$.

89. If an electrostatic field E acts on a liquid or a gaseous polar dielectric, the net dipole moment P per unit volume is

$$P(E) = \frac{e^E + e^{-E}}{e^E - e^{-E}} - \frac{1}{E}$$

Show that $\lim_{E \to 0^+} P(E) = 0$.

90. A metal cable has radius r and is covered by insulation, so that the distance from the center of the cable to the exterior of the insulation is R. The velocity v of an electrical impulse in the cable is

$$v = -c\left(\frac{r}{R}\right)^2 \ln\left(\frac{r}{R}\right)$$

where c is a positive constant. Find the following limits and interpret your answers.

(a) $\lim_{R \to r^+} v$ (b) $\lim_{r \to 0^+} v$

91. In Section 4.3 we investigated the Fresnel function $S(x) = \int_0^x \sin\left(\frac{1}{2}\pi t^2\right) dt$, which arises in the study of the diffraction of light waves. Evaluate

$$\lim_{x \to 0} \frac{S(x)}{x^3}$$

92. Suppose that the temperature in a long thin rod placed along the x-axis is initially $C/(2a)$ if $|x| \leqslant a$ and 0 if $|x| > a$. It can be shown that if the heat diffusivity of the rod is k, then the temperature of the rod at the point x at time t is

$$T(x, t) = \frac{C}{a\sqrt{4\pi kt}} \int_0^a e^{-(x-u)^2/(4kt)}\, du$$

To find the temperature distribution that results from an initial hot spot concentrated at the origin, we need to compute

$$\lim_{a \to 0} T(x, t)$$

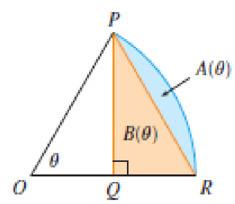
Use l'Hospital's Rule to find this limit.

93. The first appearance in print of l'Hospital's Rule was in the book *Analyse des Infiniment Petits* published by the Marquis de l'Hospital in 1696. This was the first calculus *textbook* ever published and the example that the Marquis used in that book to illustrate his rule was to find the limit of the function

$$y = \frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{aax}}{a - \sqrt[4]{ax^3}}$$

as x approaches a, where $a > 0$. (At that time it was common to write aa instead of a^2.) Solve this problem.

94. The figure shows a sector of a circle with central angle θ. Let $A(\theta)$ be the area of the segment between the chord PR and the arc PR. Let $B(\theta)$ be the area of the triangle PQR. Find $\lim_{\theta \to 0^+} A(\theta)/B(\theta)$.



95. Evaluate $\displaystyle\lim_{x \to \infty}\left[x - x^2 \ln\left(\frac{1 + x}{x}\right)\right]$.

96. Suppose f is a positive function. If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = \infty$, show that

$$\lim_{x \to a} [f(x)]^{g(x)} = 0$$

This shows that 0^∞ is not an indeterminate form.

97. If f' is continuous, $f(2) = 0$, and $f'(2) = 7$, evaluate

$$\lim_{x \to 0} \frac{f(2 + 3x) + f(2 + 5x)}{x}$$

98. For what values of a and b is the following equation true?

$$\lim_{x \to 0}\left(\frac{\sin 2x}{x^3} + a + \frac{b}{x^2}\right) = 0$$

99. If f' is continuous, use l'Hospital's Rule to show that

$$\lim_{h \to 0} \frac{f(x + h) - f(x - h)}{2h} = f'(x)$$

Explain the meaning of this equation with the aid of a diagram.

100. If f'' is continuous, show that

$$\lim_{h \to 0} \frac{f(x + h) - 2f(x) + f(x - h)}{h^2} = f''(x)$$

101. Let

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

(a) Use the definition of derivative to compute $f'(0)$.

6 Review

Concept Check

1. (a) What is a one-to-one function? How can you tell if a function is one-to-one by looking at its graph?
   (b) If f is a one-to-one function, how is its inverse function f^{-1} defined? How do you obtain the graph of f^{-1} from the graph of f?
   (c) If f is a one-to-one function and $f'(f^{-1}(a)) \neq 0$, write a formula for $(f^{-1})'(a)$.

2. (a) What are the domain and range of the natural exponential function $f(x) = e^x$?
   (b) What are the domain and range of the natural logarithmic function $f(x) = \ln x$?
   (c) How are the graphs of these functions related? Sketch these graphs by hand, using the same axes.
   (d) If a is a positive number, $a \neq 1$, write an equation that expresses $\log_a x$ in terms of $\ln x$.

3. (a) How is the inverse sine function $f(x) = \sin^{-1}x$ defined? What are its domain and range?
   (b) How is the inverse cosine function $f(x) = \cos^{-1}x$ defined? What are its domain and range?

(c) How is the inverse tangent function $f(x) = \tan^{-1} x$ defined? What are its domain and range? Sketch its graph.

4. Write the definitions of the hyperbolic functions $\sinh x$, $\cosh x$, and $\tanh x$.

5. State the derivative of each function.

(a) $y = e^x$ (b) $y = a^x$ (c) $y = \ln x$
(d) $y = \log_a x$ (e) $y = \sin^{-1} x$ (f) $y = \cos^{-1} x$
(g) $y = \tan^{-1} x$ (h) $y = \sinh x$ (i) $y = \cosh x$
(j) $y = \tanh x$ (k) $y = \sinh^{-1} x$ (l) $y = \cosh^{-1} x$
(m) $y = \tanh^{-1} x$

6. (a) How is the number e defined?
(b) Express e as a limit.
(c) Why is the natural exponential function $y = e^x$ used more often in calculus than the other exponential functions $y = a^x$?
(d) Why is the natural logarithmic function $y = \ln x$ used more often in calculus than the other logarithmic functions $y = \log_a x$?

7. (a) Write a differential equation that expresses the law of natural growth.
(b) Under what circumstances is this an appropriate model for population growth?
(c) What are the solutions of this equation?

8. (a) What does l'Hospital's Rule say?
(b) How can you use l'Hospital's Rule if you have a product $f(x)\,g(x)$ where $f(x) \to 0$ and $g(x) \to \infty$ as $x \to a$?
(c) How can you use l'Hospital's Rule if you have a difference $f(x) - g(x)$ where $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to a$?
(d) How can you use l'Hospital's Rule if you have a power $[f(x)]^{g(x)}$ where $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$?

True-False Quiz

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. If f is one-to-one, with domain $\mathbb{R}$, then $f^{-1}(f(6)) = 6$.

2. If f is one-to-one and differentiable, with domain $\mathbb{R}$, then $(f^{-1})'(6) = 1/f'(6)$.

3. The function $f(x) = \cos x$, $-\pi/2 \leqslant x \leqslant \pi/2$, is one-to-one.

4. $\tan^{-1}(-1) = 3\pi/4$

5. If $0 < a < b$, then $\ln a < \ln b$.

6. $\pi^{\sqrt{5}} = e^{\sqrt{5} \ln \pi}$

7. You can always divide by e^x.

8. If $a > 0$ and $b > 0$, then $\ln(a + b) = \ln a + \ln b$.

9. If $x > 0$, then $(\ln x)^6 = 6 \ln x$.

10. $\dfrac{d}{dx}(10^x) = x10^{x-1}$

11. $\dfrac{d}{dx}(\ln 10) = \dfrac{1}{10}$

12. The inverse function of $y = e^{3x}$ is $y = \frac{1}{3} \ln x$.

13. $\cos^{-1} x = \dfrac{1}{\cos x}$

14. $\tan^{-1} x = \dfrac{\sin^{-1} x}{\cos^{-1} x}$

15. $\cosh x \geqslant 1$ for all x

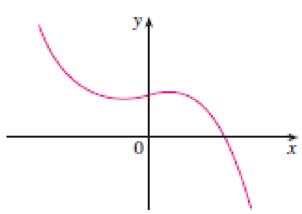
16. $\ln \dfrac{1}{10} = -\displaystyle\int_1^{10} \frac{dx}{x}$

17. $\displaystyle\int_2^{16} \frac{dx}{x} = 3 \ln 2$

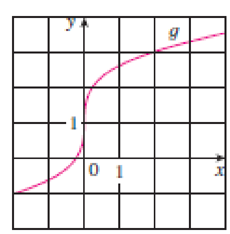
18. $\displaystyle\lim_{x \to \pi^-} \frac{\tan x}{1 - \cos x} = \lim_{x \to \pi^-} \frac{\sec^2 x}{\sin x} = \infty$

Exercises

1. The graph of f is shown. Is f one-to-one? Explain.



2. The graph of g is given.
(a) Why is g one-to-one?
(b) Estimate the value of $g^{-1}(2)$.
(c) Estimate the domain of g^{-1}.
(d) Sketch the graph of g^{-1}.



3. Suppose f is one-to-one, $f(7) = 3$, and $f'(7) = 8$. Find (a) $f^{-1}(3)$ and (b) $(f^{-1})'(3)$.

4. Find the inverse function of $f(x) = \dfrac{x+1}{2x+1}$.

5–9 Sketch a rough graph of the function without using a calculator.

5. $y = 5^x - 1$

6. $y = -e^{-x}$

7. $y = -\ln x$

8. $y = \ln(x-1)$

9. $y = 2 \arctan x$

10. Let $a > 1$. For large values of x, which of the functions $y = x^a$, $y = a^x$, and $y = \log_a x$ has the largest values and which has the smallest values?

11–12 Find the exact value of each expression.

11. (a) $e^{2\ln 3}$ (b) $\log_{10} 25 + \log_{10} 4$

12. (a) $\ln e^{\pi}$ (b) $\tan\left(\arcsin \frac{1}{2}\right)$

13–20 Solve the equation for x.

13. $\ln x = \frac{1}{3}$

14. $e^x = \frac{1}{3}$

15. $e^{e^x} = 17$

16. $\ln(1 + e^{-x}) = 3$

17. $\ln(x+1) + \ln(x-1) = 1$

18. $\log_5(c^x) = d$

19. $\tan^{-1} x = 1$

20. $\sin x = 0.3$

21–47 Differentiate.

21. $f(t) = t^2 \ln t$

22. $g(t) = \dfrac{e^t}{1 + e^t}$

23. $h(\theta) = e^{\tan 2\theta}$

24. $h(u) = 10^{\sqrt{u}}$

25. $y = \ln|\sec 5x + \tan 5x|$

26. $y = x \cos^{-1} x$

27. $y = x \tan^{-1}(4x)$

28. $y = e^{mx} \cos nx$

29. $y = \ln(\sec^2 x)$

30. $y = \sqrt{t \ln(t^4)}$

31. $y = \dfrac{e^{1/x}}{x^2}$

32. $y = (\arcsin 2x)^2$

33. $y = 3^{x \ln x}$

34. $y = e^{\cos x} + \cos(e^x)$

35. $H(v) = v \tan^{-1} v$

36. $F(z) = \log_{10}(1 + z^2)$

37. $y = x \sinh(x^2)$

38. $y = (\cos x)^x$

39. $y = \ln \sin x - \frac{1}{2} \sin^2 x$

40. $y = \arctan\left(\arcsin \sqrt{x}\right)$

41. $y = \ln\left(\dfrac{1}{x}\right) + \dfrac{1}{\ln x}$

42. $xe^y = y - 1$

43. $y = \ln(\cosh 3x)$

44. $y = \dfrac{(x^2+1)^4}{(2x+1)^3(3x-1)^5}$

45. $y = \cosh^{-1}(\sinh x)$

46. $y = x \tanh^{-1}\sqrt{x}$

47. $y = \cos\left(e^{\sqrt{\tan 3x}}\right)$

48. Show that

$$\frac{d}{dx}\left(\tfrac{1}{2}\tan^{-1}x + \tfrac{1}{4}\ln\frac{(x+1)^2}{x^2+1}\right) = \frac{1}{(1+x)(1+x^2)}$$

49–52 Find f' in terms of g'.

49. $f(x) = e^{g(x)}$

50. $f(x) = g(e^x)$

51. $f(x) = \ln|g(x)|$

52. $f(x) = g(\ln x)$

53–54 Find $f^{(n)}(x)$.

53. $f(x) = 2^x$

54. $f(x) = \ln(2x)$

55. Use mathematical induction to show that if $f(x) = xe^x$, then $f^{(n)}(x) = (x+n)e^x$.

56. Find y' if $y = x + \arctan y$.

57–58 Find an equation of the tangent to the curve at the given point.

57. $y = (2 + x)e^{-x}$, $(0, 2)$

58. $y = x \ln x$, (e, e)

59. At what point on the curve $y = [\ln(x+4)]^2$ is the tangent horizontal?

60. If $f(x) = xe^{\sin x}$, find $f'(x)$. Graph f and f' on the same screen and comment.

61. (a) Find an equation of the tangent to the curve $y = e^x$ that is parallel to the line $x - 4y = 1$.
(b) Find an equation of the tangent to the curve $y = e^x$ that passes through the origin.

62. The function $C(t) = K(e^{-at} - e^{-bt})$, where a, b, and K are positive constants and $b > a$, is used to model the concentration at time t of a drug injected into the bloodstream.
(a) Show that $\lim_{t\to\infty} C(t) = 0$.
(b) Find $C'(t)$, the rate at which the drug is cleared from circulation.
(c) When is this rate equal to 0?

63–78 Evaluate the limit.

63. $\displaystyle\lim_{x\to\infty} e^{-3x}$

64. $\displaystyle\lim_{x\to 10^-} \ln(100 - x^2)$

65. $\displaystyle\lim_{x\to 3^-} e^{2/(x-3)}$

66. $\displaystyle\lim_{x\to\infty} \arctan(x^3 - x)$

67. $\displaystyle\lim_{x\to 0^+} \ln(\sinh x)$

68. $\displaystyle\lim_{x\to\infty} e^{-x} \sin x$

69. $\displaystyle\lim_{x\to\infty} \frac{1 + 2^x}{1 - 2^x}$

70. $\displaystyle\lim_{x\to\infty} \left(1 + \frac{4}{x}\right)^x$

71. $\displaystyle\lim_{x\to 0} \frac{e^x - 1}{\tan x}$

72. $\displaystyle\lim_{x\to 0} \frac{1 - \cos x}{x^2 + x}$

73. $\lim_{x \to 0} \frac{e^{4x} - 1 - 4x}{x^2}$

74. $\lim_{x \to \infty} \frac{e^{4x} - 1 - 4x}{x^2}$

75. $\lim_{x \to -\infty} (x^2 - x^3)e^{2x}$

76. $\lim_{x \to 0^+} x^2 \ln x$

77. $\lim_{x \to 1^+} \left(\frac{x}{x-1} - \frac{1}{\ln x} \right)$

78. $\lim_{x \to (\pi/2)^-} (\tan x)^{\cos x}$

79–84 Sketch the curve using the guidelines of Section 3.5.

79. $y = e^x \sin x, \quad -\pi \leqslant x \leqslant \pi$

80. $y = \sin^{-1}(1/x)$

81. $y = x \ln x$

82. $y = e^{2x - x^2}$

83. $y = (x - 2)e^{-x}$

84. $y = x + \ln(x^2 + 1)$

85. Investigate the family of curves given by $f(x) = xe^{-cx}$, where c is a real number. Start by computing the limits as $x \to \pm\infty$. Identify any transitional values of c where the basic shape changes. What happens to the maximum or minimum points and inflection points as c changes? Illustrate by graphing several members of the family.

86. Investigate the family of functions $f(x) = cxe^{-cx^2}$. What happens to the maximum and minimum points and the inflection points as c changes? Illustrate your conclusions by graphing several members of the family.

87. An equation of motion of the form $s = Ae^{-ct}\cos(\omega t + \delta)$ represents damped oscillation of an object. Find the velocity and acceleration of the object.

88. (a) Show that there is exactly one root of the equation $\ln x = 3 - x$ and that it lies between 2 and e.
(b) Find the root of the equation in part (a) correct to four decimal places.

89. A bacteria culture contains 200 cells initially and grows at a rate proportional to its size. After half an hour the population has increased to 360 cells.
(a) Find the number of bacteria after t hours.
(b) Find the number of bacteria after 4 hours.
(c) Find the rate of growth after 4 hours.
(d) When will the population reach 10,000?

90. Cobalt-60 has a half-life of 5.24 years.
(a) Find the mass that remains from a 100-mg sample after 20 years.
(b) How long would it take for the mass to decay to 1 mg?

91. The biologist G. F. Gause conducted an experiment in the 1930s with the protozoan *Paramecium* and used the population function

$$P(t) = \frac{64}{1 + 31e^{-0.7944t}}$$

to model his data, where t was measured in days. Use this model to determine when the population was increasing most rapidly.

92–105 Evaluate the integral.

92. $\int_0^4 \frac{1}{16 + t^2}\, dt$

93. $\int_0^1 ye^{-2y^2}\, dy$

94. $\int_2^5 \frac{dr}{1 + 2r}$

95. $\int_0^1 \frac{e^x}{1 + e^{2x}}\, dx$

96. $\int_0^{\pi/2} \frac{\cos x}{1 + \sin^2 x}\, dx$

97. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$

98. $\int \frac{\cos(\ln x)}{x}\, dx$

99. $\int \frac{x + 1}{x^2 + 2x}\, dx$

100. $\int \frac{\csc^2 x}{1 + \cot x}\, dx$

101. $\int \tan x \ln(\cos x)\, dx$

102. $\int \frac{x}{\sqrt{1 - x^4}}\, dx$

103. $\int 2^{\tan \theta} \sec^2\theta\, d\theta$

104. $\int \sinh au\, du$

105. $\int \left(\frac{1 - x}{x} \right)^2 dx$

106–108 Use properties of integrals to prove the inequality.

106. $\int_0^1 \sqrt{1 + e^{2x}}\, dx \geqslant e - 1$

107. $\int_0^1 e^x \cos x\, dx \leqslant e - 1$

108. $\int_0^1 x \sin^{-1} x\, dx \leqslant \pi/4$

109–110 Find $f'(x)$.

109. $f(x) = \int_1^{\sqrt{x}} \frac{e^s}{s}\, ds$

110. $f(x) = \int_{\ln x}^{2x} e^{-t^2}\, dt$

111. Find the average value of the function $f(x) = 1/x$ on the interval $[1, 4]$.

112. Find the area of the region bounded by the curves $y = e^x$, $y = e^{-x}$, $x = -2$, and $x = 1$.

113. Find the volume of the solid obtained by rotating about the y-axis the region under the curve $y = 1/(1 + x^4)$ from $x = 0$ to $x = 1$.

114. If $f(x) = x + x^2 + e^x$, find $(f^{-1})'(1)$.

115. If $f(x) = \ln x + \tan^{-1}x$, find $(f^{-1})'(\pi/4)$.

116. What is the area of the largest rectangle in the first quadrant with two sides on the axes and one vertex on the curve $y = e^{-x}$?

117. What is the area of the largest triangle in the first quadrant with two sides on the axes and the third side tangent to the curve $y = e^{-x}$?