6.6 Exercises

1–10 Find the exact value of each expression.

1. (a) $\sin^{-1}(0.5)$ (b) $\cos^{-1}(-1)$

2. (a) $\tan^{-1}\sqrt{3}$ (b) $\sec^{-1} 2$

3. (a) $\csc^{-1}\sqrt{2}$ (b) $\sin^{-1}(1/\sqrt{2})$

4. (a) $\cot^{-1}(-\sqrt{3})$ (b) $\arcsin 1$

5. (a) $\tan(\arctan 10)$ (b) $\sin^{-1}(\sin(7\pi/3))$

6. (a) $\tan^{-1}(\tan 3\pi/4)$ (b) $\cos\left(\arcsin \frac{1}{2}\right)$

7. $\tan\left(\sin^{-1}\left(\frac{2}{3}\right)\right)$

8. $\csc\left(\arccos \frac{3}{5}\right)$

9. $\sin\left(2 \tan^{-1}\sqrt{2}\right)$

10. $\cos\left(\tan^{-1} 2 + \tan^{-1} 3\right)$

11. Prove that $\cos(\sin^{-1} x) = \sqrt{1 - x^2}$.

12–14 Simplify the expression.

12. $\tan(\sin^{-1} x)$

13. $\sin(\tan^{-1} x)$

14. $\cos(2 \tan^{-1} x)$

15–16 Graph the given functions on the same screen. How are these graphs related?

15. $y = \sin x, \; -\pi/2 \leqslant x \leqslant \pi/2; \quad y = \sin^{-1} x; \quad y = x$

16. $y = \tan x, \; -\pi/2 < x < \pi/2; \quad y = \tan^{-1} x; \quad y = x$

17. Prove Formula 6 for the derivative of $\cos^{-1}$ by the same method as for Formula 3.

18. (a) Prove that $\sin^{-1}x + \cos^{-1}x = \pi/2$.
(b) Use part (a) to prove Formula 6.

19. Prove that $\dfrac{d}{dx}(\cot^{-1}x) = -\dfrac{1}{1 + x^2}$.

20. Prove that $\dfrac{d}{dx}(\sec^{-1}x) = \dfrac{1}{x\sqrt{x^2 - 1}}$.

21. Prove that $\dfrac{d}{dx}(\csc^{-1}x) = -\dfrac{1}{x\sqrt{x^2 - 1}}$.

22–35 Find the derivative of the function. Simplify where possible.

22. $y = \tan^{-1}(x^2)$

23. $y = (\tan^{-1}x)^2$

24. $y = \cos^{-1}(\sin^{-1}t)$

25. $y = \sin^{-1}(2x + 1)$

26. $g(x) = \sqrt{x^2 - 1}\sec^{-1}x$

27. $y = x\sin^{-1}x + \sqrt{1 - x^2}$

28. $F(\theta) = \arcsin\sqrt{\sin\theta}$

29. $y = \cos^{-1}(e^{2x})$

30. $y = \arctan\sqrt{\dfrac{1 - x}{1 + x}}$

31. $y = \arctan(\cos\theta)$

32. $y = \tan^{-1}\left(x - \sqrt{1 + x^2}\right)$

33. $h(t) = \cot^{-1}(t) + \cot^{-1}(1/t)$

34. $y = \tan^{-1}\left(\dfrac{x}{a}\right) + \ln\sqrt{\dfrac{x - a}{x + a}}$

35. $y = \arccos\left(\dfrac{b + a\cos x}{a + b\cos x}\right), \quad 0 \leq x \leq \pi, \ a > b > 0$

36–37 Find the derivative of the function. Find the domains of the function and its derivative.

36. $f(x) = \arcsin(e^x)$

37. $g(x) = \cos^{-1}(3 - 2x)$

38. Find y' if $\tan^{-1}(x^2y) = x + xy^2$.

39. If $g(x) = x\sin^{-1}(x/4) + \sqrt{16 - x^2}$, find $g'(2)$.

40. Find an equation of the tangent line to the curve $y = 3\arccos(x/2)$ at the point $(1, \pi)$.

41–42 Find $f'(x)$. Check that your answer is reasonable by comparing the graphs of f and f'.

41. $f(x) = \sqrt{1 - x^2}\arcsin x$

42. $f(x) = \arctan(x^2 - x)$

43–46 Find the limit.

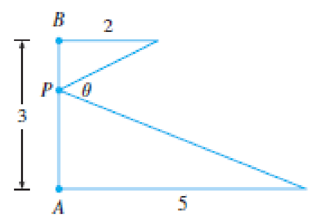
43. $\lim_{x \to -1^+} \sin^{-1}x$

44. $\lim_{x \to \infty} \arccos\left(\dfrac{1 + x^2}{1 + 2x^2}\right)$

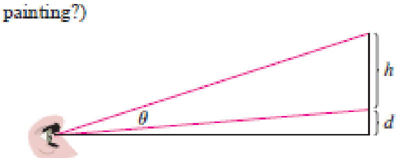
45. $\lim_{x \to \infty} \arctan(e^x)$

46. $\lim_{x \to 0^+} \tan^{-1}(\ln x)$

47. Where should the point P be chosen on the line segment AB so as to maximize the angle θ?



48. A painting in an art gallery has height h and is hung so that its lower edge is a distance d above the eye of an observer (as in the figure). How far from the wall should the observer stand to get the best view? (In other words, where should the observer stand so as to maximize the angle θ subtended at her eye by the painting?)



49. A ladder 10 ft long leans against a vertical wall. If the bottom of the ladder slides away from the base of the wall at a speed of 2 ft/s, how fast is the angle between the ladder and the wall changing when the bottom of the ladder is 6 ft from the base of the wall?

50. A lighthouse is located on a small island, 3 km away from the nearest point P on a straight shoreline, and its light makes four revolutions per minute. How fast is the beam of light moving along the shoreline when it is 1 km from P?

51–54 Sketch the curve using the guidelines of Section 3.5.

51. $y = \sin^{-1}\left(\dfrac{x}{x + 1}\right)$

52. $y = \tan^{-1}\left(\dfrac{x - 1}{x + 1}\right)$

53. $y = x - \tan^{-1}x$

54. $y = \tan^{-1}(\ln x)$

CAS **55.** If $f(x) = \arctan(\cos(3\arcsin x))$, use the graphs of f, f', and f'' to estimate the x-coordinates of the maximum and minimum points and inflection points of f.

56. Investigate the family of curves given by $f(x) = x - c\sin^{-1}x$. What happens to the number of maxima and minima as c changes? Graph several members of the family to illustrate what you discover.

57. Find the most general antiderivative of the function

$$f(x) = \frac{2 + x^2}{1 + x^2}$$

58. Find $f(x)$ if $f'(x) = 4/\sqrt{1 - x^2}$ and $f(\frac{1}{2}) = 1$.

59–70 Evaluate the integral.

59. $\displaystyle\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{8}{1 + x^2}\,dx$

60. $\displaystyle\int_{1/2}^{1/\sqrt{2}} \frac{4}{\sqrt{1 - x^2}}\,dx$

61. $\displaystyle\int_0^{1/2} \frac{\sin^{-1}x}{\sqrt{1 - x^2}}\,dx$

62. $\displaystyle\int_0^{\sqrt{3}/4} \frac{dx}{1 + 16x^2}$

63. $\displaystyle\int \frac{1 + x}{1 + x^2}\,dx$

64. $\displaystyle\int_0^{\pi/2} \frac{\sin x}{1 + \cos^2 x}\,dx$

65. $\displaystyle\int \frac{dx}{\sqrt{1 - x^2}\,\sin^{-1}x}$

66. $\displaystyle\int \frac{1}{x\sqrt{x^2 - 4}}\,dx$

67. $\displaystyle\int \frac{t^2}{\sqrt{1 - t^6}}\,dt$

68. $\displaystyle\int \frac{e^{2x}}{\sqrt{1 - e^{4x}}}\,dx$

69. $\displaystyle\int \frac{dx}{\sqrt{x}\,(1 + x)}$

70. $\displaystyle\int \frac{x}{1 + x^4}\,dx$

71. Use the method of Example 8 to show that, if $a > 0$,

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C$$

72. The region under the curve $y = 1/\sqrt{x^2 + 4}$ from $x = 0$ to $x = 2$ is rotated about the x-axis. Find the volume of the resulting solid.

73. Evaluate $\int_0^1 \sin^{-1}x\,dx$ by interpreting it as an area and integrating with respect to y instead of x.

74. Prove that, for $xy \neq 1$,

$$\arctan x + \arctan y = \arctan\frac{x + y}{1 - xy}$$

if the left side lies between $-\pi/2$ and $\pi/2$.

75. Use the result of Exercise 74 to prove the following:
(a) $\arctan\frac{1}{2} + \arctan\frac{1}{3} = \pi/4$
(b) $2\arctan\frac{1}{3} + \arctan\frac{1}{7} = \pi/4$

76. (a) Sketch the graph of the function $f(x) = \sin(\sin^{-1}x)$.
(b) Sketch the graph of the function $g(x) = \sin^{-1}(\sin x)$, $x \in \mathbb{R}$.
(c) Show that $g'(x) = \dfrac{\cos x}{|\cos x|}$.
(d) Sketch the graph of $h(x) = \cos^{-1}(\sin x)$, $x \in \mathbb{R}$, and find its derivative.

77. Use the method of Example 6 to prove the identity

$$2\sin^{-1}x = \cos^{-1}(1 - 2x^2) \qquad x \geqslant 0$$

78. Prove the identity

$$\arcsin\frac{x - 1}{x + 1} = 2\arctan\sqrt{x} - \frac{\pi}{2}$$

79. Some authors define $y = \sec^{-1}x \iff \sec y = x$ and $y \in [0, \pi/2) \cup (\pi/2, \pi]$. Show that with this definition we have (instead of the formula given in Exercise 20)

$$\frac{d}{dx}(\sec^{-1}x) = \frac{1}{|x|\sqrt{x^2 - 1}} \qquad |x| > 1$$

80. Let $f(x) = x\arctan(1/x)$ if $x \neq 0$ and $f(0) = 0$.
(a) Is f continuous at 0?
(b) Is f differentiable at 0?