6.3 Exercises

1. (a) How is the logarithmic function $y = \log_a x$ defined?
   (b) What is the domain of this function?
   (c) What is the range of this function?
   (d) Sketch the general shape of the graph of the function $y = \log_a x$ if $a > 1$.

2. (a) What is the natural logarithm?
   (b) What is the common logarithm?
   (c) Sketch the graphs of the natural logarithm function and the natural exponential function with a common set of axes.

3–8 Find the exact value of each expression.

3. (a) $\log_5 125$ (b) $\log_3\left(\frac{1}{27}\right)$
4. (a) $\ln(1/e)$ (b) $\log_{10}\sqrt{10}$
5. (a) $e^{\ln 4.5}$ (b) $\log_{10} 0.0001$
6. (a) $\log_{1.5} 2.25$ (b) $\log_5 4 - \log_5 500$
7. (a) $\log_2 6 - \log_2 15 + \log_2 20$
   (b) $\log_3 100 - \log_3 18 - \log_3 50$
8. (a) $e^{-2\ln 5}$ (b) $\ln\left(\ln e^{e^{10}}\right)$

9–12 Use the properties of logarithms to expand the quantity.

9. $\ln\sqrt{ab}$
10. $\log_{10}\sqrt{\dfrac{x-1}{x+1}}$
11. $\ln\dfrac{x^2}{y^3z^4}$
12. $\ln\left(s^4\sqrt{t\sqrt{u}}\right)$

13–18 Express the quantity as a single logarithm.

13. $2\ln x + 3\ln y - \ln z$
14. $\log_{10} 4 + \log_{10} a - \frac{1}{3}\log_{10}(a+1)$
15. $\ln 5 + 5\ln 3$
16. $\ln 3 + \frac{1}{3}\ln 8$
17. $\frac{1}{3}\ln(x+2)^3 + \frac{1}{2}[\ln x - \ln(x^2 + 3x + 2)^2]$
18. $\ln(a+b) + \ln(a-b) - 2\ln c$

19. Use Formula 7 to evaluate each logarithm correct to six decimal places.
    (a) $\log_{12} e$ (b) $\log_6 13.54$ (c) $\log_2 \pi$

20–22 Use Formula 7 to graph the given functions on a common screen. How are these graphs related?

20. $y = \log_2 x$, $y = \log_4 x$, $y = \log_6 x$, $y = \log_8 x$

21. $y = \log_{1.5} x$, $y = \ln x$, $y = \log_{10} x$, $y = \log_{50} x$

22. $y = \ln x$, $y = \log_{10} x$, $y = e^x$, $y = 10^x$

23–24 Make a rough sketch of the graph of each function. Do not use a calculator. Just use the graphs given in Figures 2 and 3 and, if necessary, the transformations of Section 1.3.

23. (a) $y = \log_{10}(x + 5)$ (b) $y = -\ln x$

24. (a) $y = \ln(-x)$ (b) $y = \ln |x|$

25–26

(a) What are the domain and range of f?

(b) What is the x-intercept of the graph of f?

(c) Sketch the graph of f.

25. $f(x) = \ln x + 2$

26. $f(x) = \ln(x - 1) - 1$

27–36 Solve each equation for x.

27. (a) $e^{7-4x} = 6$ (b) $\ln(3x - 10) = 2$

28. (a) $\ln(x^2 - 1) = 3$ (b) $e^{2x} - 3e^x + 2 = 0$

29. (a) $2^{x-5} = 3$ (b) $\ln x + \ln(x - 1) = 1$

30. (a) $e^{3x+1} = k$ (b) $\log_2(mx) = c$

31. $e - e^{-2x} = 1$

32. $10(1 + e^{-x})^{-1} = 3$

33. $\ln(\ln x) = 1$

34. $e^{e^x} = 10$

35. $e^{2x} - e^x - 6 = 0$

36. $\ln(2x + 1) = 2 - \ln x$

37–38 Find the solution of the equation correct to four decimal places.

37. (a) $e^{2+5x} = 100$ (b) $\ln(e^x - 2) = 3$

38. (a) $\ln(1 + \sqrt{x}) = 2$ (b) $3^{1/(x-4)} = 7$

39–40 Solve each inequality for x.

39. (a) $\ln x < 0$ (b) $e^x > 5$

40. (a) $1 < e^{3x-1} < 2$ (b) $1 - 2\ln x < 3$

41. Suppose that the graph of $y = \log_2 x$ is drawn on a coordinate grid where the unit of measurement is an inch. How many miles to the right of the origin do we have to move before the height of the curve reaches 3 ft?

42. The velocity of a particle that moves in a straight line under the influence of viscous forces is $v(t) = ce^{-kt}$, where c and k are positive constants.

(a) Show that the acceleration is proportional to the velocity.

(b) Explain the significance of the number c.

(c) At what time is the velocity equal to half the initial velocity?

43. The geologist C. F. Richter defined the magnitude of an earthquake to be $\log_{10}(I/S)$, where I is the intensity of the quake (measured by the amplitude of a seismograph 100 km from the epicenter) and S is the intensity of a "standard" earthquake (where the amplitude is only 1 micron $= 10^{-4}$ cm). The 1989 Loma Prieta earthquake that shook San Francisco had a magnitude of 7.1 on the Richter scale. The 1906 San Francisco earthquake was 16 times as intense. What was its magnitude on the Richter scale?

44. A sound so faint that it can just be heard has intensity $I_0 = 10^{-12}$ watt/m^2 at a frequency of 1000 hertz (Hz). The loudness, in decibels (dB), of a sound with intensity I is then defined to be $L = 10\log_{10}(I/I_0)$. Amplified rock music is measured at 120 dB, whereas the noise from a motor-driven lawn mower is measured at 106 dB. Find the ratio of the intensity of the rock music to that of the mower.

45. If a bacteria population starts with 100 bacteria and doubles every three hours, then the number of bacteria after t hours is $n = f(t) = 100 \cdot 2^{t/3}$.

(a) Find the inverse of this function and explain its meaning.

(b) When will the population reach 50,000?

46. When a camera flash goes off, the batteries immediately begin to recharge the flash's capacitor, which stores electric charge given by

$$Q(t) = Q_0(1 - e^{-t/a})$$

(The maximum charge capacity is Q_0 and t is measured in seconds.)

(a) Find the inverse of this function and explain its meaning.

(b) How long does it take to recharge the capacitor to 90% of capacity if $a = 2$?

47–52 Find the limit.

47. $\lim_{x \to 3^+} \ln(x^2 - 9)$

48. $\lim_{x \to 2^-} \log_5(8x - x^4)$

49. $\lim_{x \to 0} \ln(\cos x)$

50. $\lim_{x \to 0^+} \ln(\sin x)$

51. $\lim_{x \to \infty} [\ln(1 + x^2) - \ln(1 + x)]$

52. $\lim_{x \to \infty} [\ln(2 + x) - \ln(1 + x)]$

53–54 Find the domain of the function.

53. $f(x) = \log_{10}(x^2 - 9)$

54. $f(x) = \ln x + \ln(2 - x)$

55–57 Find (a) the domain of f and (b) f^{-1} and its domain.

55. $f(x) = \sqrt{3 - e^{2x}}$

56. $f(x) = \ln(2 + \ln x)$

57. $f(x) = \ln(e^x - 3)$

58. (a) What are the values of $e^{\ln 300}$ and $\ln(e^{300})$?
(b) Use your calculator to evaluate $e^{\ln 300}$ and $\ln(e^{300})$. What do you notice? Can you explain why the calculator has trouble?

59–64 Find the inverse function.

59. $y = \ln(x + 3)$

60. $y = 2^{10^x}$

61. $f(x) = e^{x^3}$

62. $y = (\ln x)^2, \quad x \geqslant 1$

63. $y = \log_{10}\left(1 + \dfrac{1}{x}\right)$

64. $y = \dfrac{e^x}{1 + 2e^x}$

65. On what interval is the function $f(x) = e^{3x} - e^x$ increasing?

66. On what interval is the curve $y = 2e^x - e^{-3x}$ concave downward?

67. (a) Show that the function $f(x) = \ln\left(x + \sqrt{x^2 + 1}\right)$ is an odd function.
(b) Find the inverse function of f.

68. Find an equation of the tangent to the curve $y = e^{-x}$ that is perpendicular to the line $2x - y = 8$.

69. Show that the equation $x^{1/\ln x} = 2$ has no solution. What can you say about the function $f(x) = x^{1/\ln x}$?

70. Any function of the form $f(x) = [g(x)]^{h(x)}$, where $g(x) > 0$, can be analyzed as a power of e by writing $g(x) = e^{\ln g(x)}$ so that $f(x) = e^{h(x)\ln g(x)}$. Using this device, calculate each limit.
(a) $\lim_{x \to \infty} x^{\ln x}$
(b) $\lim_{x \to 0^+} x^{-\ln x}$
(c) $\lim_{x \to 0^+} x^{1/x}$
(d) $\lim_{x \to \infty} (\ln 2x)^{-\ln x}$

71. Let $a > 1$. Prove, using Definitions 3.4.6 and 3.4.7, that
(a) $\lim_{x \to -\infty} a^x = 0$
(b) $\lim_{x \to \infty} a^x = \infty$

72. (a) Compare the rates of growth of $f(x) = x^{0.1}$ and $g(x) = \ln x$ by graphing both f and g in several viewing rectangles. When does the graph of f finally surpass the graph of g?
(b) Graph the function $h(x) = (\ln x)/x^{0.1}$ in a viewing rectangle that displays the behavior of the function as $x \to \infty$.
(c) Find a number N such that

$$\text{if} \quad x > N \quad \text{then} \quad \frac{\ln x}{x^{0.1}} < 0.1$$

73. Solve the inequality $\ln(x^2 - 2x - 2) \leqslant 0$.

74. A **prime number** is a positive integer that has no factors other than 1 and itself. The first few primes are 2, 3, 5, 7, 11, 13, 17, ... We denote by $\pi(n)$ the number of primes that are less than or equal to n. For instance, $\pi(15) = 6$ because there are six primes smaller than 15.
(a) Calculate the numbers $\pi(25)$ and $\pi(100)$.
[*Hint:* To find $\pi(100)$, first compile a list of the primes up to 100 using the *sieve of Eratosthenes:* Write the numbers from 2 to 100 and cross out all multiples of 2. Then cross out all multiples of 3. The next remaining number is 5, so cross out all remaining multiples of it, and so on.]
(b) By inspecting tables of prime numbers and tables of logarithms, the great mathematician K. F. Gauss made the guess in 1792 (when he was 15) that the number of primes up to n is approximately $n/\ln n$ when n is large. More precisely, he conjectured that

$$\lim_{n \to \infty} \frac{\pi(n)}{n/\ln n} = 1$$

This was finally proved, a hundred years later, by Jacques Hadamard and Charles de la Vallée Poussin and is called the **Prime Number Theorem**. Provide evidence for the truth of this theorem by computing the ratio of $\pi(n)$ to $n/\ln n$ for $n = 100, 1000, 10^4, 10^5, 10^6$, and 10^7. Use the following data: $\pi(1000) = 168$, $\pi(10^4) = 1229$, $\pi(10^5) = 9592$, $\pi(10^6) = 78{,}498$, $\pi(10^7) = 664{,}579$.
(c) Use the Prime Number Theorem to estimate the number of primes up to a billion.