6.2 Exercises

1. (a) Write an equation that defines the exponential function with base $a > 0$.
   (b) What is the domain of this function?
   (c) If $a \neq 1$, what is the range of this function?
   (d) Sketch the general shape of the graph of the exponential function for each of the following cases.
   (i) $a > 1$ (ii) $a = 1$ (iii) $0 < a < 1$

2. (a) How is the number e defined?
   (b) What is an approximate value for e?
   (c) What is the natural exponential function?

3–6 Graph the given functions on a common screen. How are these graphs related?

3. $y = 2^x, \quad y = e^x, \quad y = 5^x, \quad y = 20^x$

4. $y = e^x, \quad y = e^{-x}, \quad y = 8^x, \quad y = 8^{-x}$

5. $y = 3^x, \quad y = 10^x, \quad y = \left(\frac{1}{3}\right)^x, \quad y = \left(\frac{1}{10}\right)^x$

6. $y = 0.9^x, \quad y = 0.6^x, \quad y = 0.3^x, \quad y = 0.1^x$

7–12 Make a rough sketch of the graph of the function. Do not use a calculator. Just use the graphs given in Figures 3 and 12 and, if necessary, the transformations of Section 1.3.

7. $y = 10^{x+2}$

8. $y = (0.5)^x - 2$

9. $y = -2^{-x}$

10. $y = e^{|x|}$

11. $y = 1 - \frac{1}{2}e^{-x}$

12. $y = 2(1 - e^x)$

13. Starting with the graph of $y = e^x$, write the equation of the graph that results from
    (a) shifting 2 units downward
    (b) shifting 2 units to the right
    (c) reflecting about the x-axis
    (d) reflecting about the y-axis
    (e) reflecting about the x-axis and then about the y-axis

14. Starting with the graph of $y = e^x$, find the equation of the graph that results from
    (a) reflecting about the line $y = 4$
    (b) reflecting about the line $x = 2$

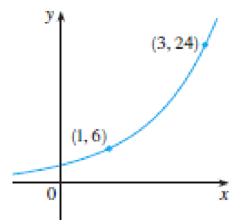
15–16 Find the domain of each function.

15. (a) $f(x) = \dfrac{1 - e^{x^2}}{1 - e^{1-x^2}}$ (b) $f(x) = \dfrac{1 + x}{e^{\cos x}}$

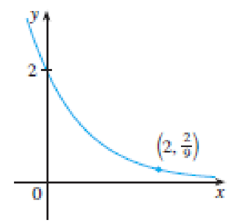
16. (a) $g(t) = \sin(e^{-t})$ (b) $g(t) = \sqrt{1 - 2^t}$

17–18 Find the exponential function $f(x) = Ca^x$ whose graph is given.

17.



18.



19. Suppose the graphs of $f(x) = x^2$ and $g(x) = 2^x$ are drawn on a coordinate grid where the unit of measurement is 1 inch. Show that, at a distance 2 ft to the right of the origin, the height of the graph of f is 48 ft but the height of the graph of g is about 265 mi.

20. Compare the functions $f(x) = x^5$ and $g(x) = 5^x$ by graphing both functions in several viewing rectangles. Find all points of intersection of the graphs correct to one decimal place. Which function grows more rapidly when x is large?

21. Compare the functions $f(x) = x^{10}$ and $g(x) = e^x$ by graphing both f and g in several viewing rectangles. When does the graph of g finally surpass the graph of f?

22. Use a graph to estimate the values of x such that $e^x > 1{,}000{,}000{,}000$.

23–30 Find the limit.

23. $\displaystyle\lim_{x \to \infty} (1.001)^x$

24. $\displaystyle\lim_{x \to -\infty} (1.001)^x$

25. $\displaystyle\lim_{x \to \infty} \frac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}}$

26. $\displaystyle\lim_{x \to \infty} e^{-x^2}$

27. $\displaystyle\lim_{x \to 2^+} e^{3/(2-x)}$

28. $\displaystyle\lim_{x \to 2^-} e^{3/(2-x)}$

29. $\displaystyle\lim_{x \to \infty} (e^{-2x} \cos x)$

30. $\displaystyle\lim_{x \to (\pi/2)^+} e^{\tan x}$

31–50 Differentiate the function.

31. $f(x) = e^5$

32. $k(r) = e^r + r^e$

33. $f(x) = (x^3 + 2x)e^x$

34. $y = \dfrac{e^x}{1 - e^x}$

35. $y = e^{ax^3}$

36. $y = e^{-2t} \cos 4t$

37. $y = xe^{-kx}$

38. $y = \dfrac{1}{s + ke^x}$

39. $f(u) = e^{1/u}$

40. $f(t) = \sin(e^t) + e^{\sin t}$

41. $F(t) = e^{t \sin 2t}$

42. $y = x^2 e^{-1/x}$

43. $y = \sqrt{1 + 2e^{3x}}$

44. $y = e^{k \tan \sqrt{x}}$

45. $y = e^{e^x}$

46. $y = \dfrac{e^u - e^{-u}}{e^u + e^{-u}}$

47. $y = \dfrac{ae^x + b}{ce^x + d}$

48. $y = \sqrt{1 + xe^{-2x}}$

49. $y = \cos\left(\dfrac{1 - e^{2x}}{1 + e^{2x}}\right)$

50. $f(t) = \sin^2(e^{\sin^2 t})$

51–52 Find an equation of the tangent line to the curve at the given point.

51. $y = e^{2x} \cos \pi x, \quad (0, 1)$

52. $y = \dfrac{e^x}{x}, \quad (1, e)$

53. Find y' if $e^{x/y} = x - y$.

54. Find an equation of the tangent line to the curve $xe^y + ye^x = 1$ at the point $(0, 1)$.

55. Show that the function $y = e^x + e^{-x/2}$ satisfies the differential equation $2y'' - y' - y = 0$.

56. Show that the function $y = Ae^{-x} + Bxe^{-x}$ satisfies the differential equation $y'' + 2y' + y = 0$.

57. For what values of r does the function $y = e^{rx}$ satisfy the equation $y'' + 6y' + 8y = 0$?

58. Find the values of λ for which $y = e^{\lambda x}$ satisfies the equation $y + y' = y''$.

59. If $f(x) = e^{2x}$, find a formula for $f^{(n)}(x)$.

60. Find the thousandth derivative of $f(x) = xe^{-x}$.

61. (a) Use the Intermediate Value Theorem to show that there is a root of the equation $e^x + x = 0$.
(b) Use Newton's method to find the root of the equation in part (a) correct to six decimal places.

62. Use a graph to find an initial approximation (to one decimal place) to the root of the equation $4e^{-x^2} \sin x = x^2 - x + 1$. Then use Newton's method to find the root correct to eight decimal places.

63. Under certain circumstances a rumor spreads according to the equation

$$p(t) = \frac{1}{1 + ae^{-kt}}$$

where $p(t)$ is the proportion of the population that knows the rumor at time t and a and k are positive constants. [In Section 9.4 we will see that this is a reasonable model for $p(t)$.]
(a) Find $\lim_{t \to \infty} p(t)$.
(b) Find the rate of spread of the rumor.
(c) Graph p for the case $a = 10$, $k = 0.5$ with t measured in hours. Use the graph to estimate how long it will take for 80% of the population to hear the rumor.

64. An object is attached to the end of a vibrating spring and its displacement from its equilibrium position is $y = 8e^{-t/2} \sin 4t$, where t is measured in seconds and y is measured in centimeters.
(a) Graph the displacement function together with the functions $y = 8e^{-t/2}$ and $y = -8e^{-t/2}$. How are these graphs related? Can you explain why?
(b) Use the graph to estimate the maximum value of the displacement. Does it occur when the graph touches the graph of $y = 8e^{-t/2}$?
(c) What is the velocity of the object when it first returns to its equilibrium position?
(d) Use the graph to estimate the time after which the displacement is no more than 2 cm from equilibrium.

65. Find the absolute maximum value of the function $f(x) = x - e^x$.

66. Find the absolute minimum value of the function $g(x) = e^x/x$, $x > 0$.

67–68 Find the absolute maximum and absolute minimum values of f on the given interval.

67. $f(x) = xe^{-x^2/8}, \quad [-1, 4]$

68. $f(x) = x^2 e^{-x/2}, \quad [-1, 6]$

69–70 Find (a) the intervals of increase or decrease, (b) the intervals of concavity, and (c) the points of inflection.

69. $f(x) = (1 - x)e^{-x}$

70. $f(x) = \dfrac{e^x}{x^2}$

71–73 Discuss the curve using the guidelines of Section 3.5.

71. $y = e^{-1/(x+1)}$

72. $y = e^{-x} \sin x, \quad 0 \leq x \leq 2\pi$

73. $y = 1/(1 + e^{-x})$

74. Let $g(x) = e^{cx} + f(x)$ and $h(x) = e^{kx} f(x)$, where $f(0) = 3$, $f'(0) = 5$, and $f''(0) = -2$.
(a) Find $g'(0)$ and $g''(0)$ in terms of c.
(b) In terms of k, find an equation of the tangent line to the graph of h at the point where $x = 0$.

75. A *drug response curve* describes the level of medication in the bloodstream after a drug is administered. A surge function $S(t) = At^p e^{-kt}$ is often used to model the response curve, reflecting an initial surge in the drug level and then a more gradual decline. If, for a particular drug, $A = 0.01$, $p = 4$, $k = 0.07$, and t is measured in minutes, estimate the times corresponding to the inflection points and explain their significance. If you have a graphing device, use it to graph the drug response curve.

76–77 Draw a graph of f that shows all the important aspects of the curve. Estimate the local maximum and minimum values and then use calculus to find these values exactly. Use a graph of f'' to estimate the inflection points.

76. $f(x) = e^{\cos x}$ **77.** $f(x) = e^{x^3 - x}$

78. The family of bell-shaped curves

$$y = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$$

occurs in probability and statistics, where it is called the *normal density function*. The constant μ is called the *mean* and the positive constant σ is called the *standard deviation*. For simplicity, let's scale the function so as to remove the factor $1/(\sigma\sqrt{2\pi})$ and let's analyze the special case where $\mu = 0$. So we study the function $f(x) = e^{-x^2/(2\sigma^2)}$.

(a) Find the asymptote, maximum value, and inflection points of f.

(b) What role does σ play in the shape of the curve?

(c) Illustrate by graphing four members of this family on the same screen.

79–90 Evaluate the integral.

79. $\int_0^1 (x^e + e^x)\,dx$ **80.** $\int_{-5}^5 e\,dx$

81. $\int_0^2 \frac{dx}{e^{\pi x}}$ **82.** $\int x^2 e^{x^3} dx$

83. $\int e^x\sqrt{1 + e^x}\,dx$ **84.** $\int \frac{(1 + e^x)^2}{e^x}\,dx$

85. $\int (e^x + e^{-x})^2\,dx$ **86.** $\int e^x(4 + e^x)^5\,dx$

87. $\int e^{\tan x}\sec^2 x\,dx$ **88.** $\int e^x\cos(e^x)\,dx$

89. $\int_1^2 \frac{e^{1/x}}{x^2}\,dx$ **90.** $\int_0^1 \frac{\sqrt{1 + e^{-x}}}{e^x}\,dx$

91. Find, correct to three decimal places, the area of the region bounded by the curves $y = e^x$, $y = e^{3x}$, and $x = 1$.

92. Find $f(x)$ if $f''(x) = 3e^x + 5\sin x$, $f(0) = 1$, and $f'(0) = 2$.

93. Find the volume of the solid obtained by rotating about the x-axis the region bounded by the curves $y = e^x$, $y = 0$, $x = 0$, and $x = 1$.

94. Find the volume of the solid obtained by rotating about the y-axis the region bounded by the curves $y = e^{-x^2}$, $y = 0$, $x = 0$, and $x = 1$.

95. The **error function**

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$$

is used in probability, statistics, and engineering. Show that $\int_a^b e^{-t^2}\,dt = \frac{1}{2}\sqrt{\pi}\,[\text{erf}(b) - \text{erf}(a)]$.

96. Show that the function

$$y = e^{x^2}\text{erf}(x)$$

satisfies the differential equation

$$y' = 2xy + 2/\sqrt{\pi}$$

97. An oil storage tank ruptures at time $t = 0$ and oil leaks from the tank at a rate of $r(t) = 100e^{-0.01t}$ liters per minute. How much oil leaks out during the first hour?

98. A bacteria population starts with 400 bacteria and grows at a rate of $r(t) = (450.268)e^{1.12567t}$ bacteria per hour. How many bacteria will there be after three hours?

99. If $f(x) = 3 + x + e^x$, find $(f^{-1})'(4)$.

100. Evaluate $\lim_{x \to \pi} \frac{e^{\sin x} - 1}{x - \pi}$.

101. If you graph the function

$$f(x) = \frac{1 - e^{1/x}}{1 + e^{1/x}}$$

you'll see that f appears to be an odd function. Prove it.

102. Graph several members of the family of functions

$$f(x) = \frac{1}{1 + ae^{bx}}$$

where $a > 0$. How does the graph change when b changes? How does it change when a changes?

103. (a) Show that $e^x \geqslant 1 + x$ if $x \geqslant 0$.
[*Hint:* Show that $f(x) = e^x - (1 + x)$ is increasing for $x > 0$.]

(b) Deduce that $\frac{4}{3} \leqslant \int_0^1 e^{x^2}\,dx \leqslant e$.

104. (a) Use the inequality of Exercise 103(a) to show that, for $x \geqslant 0$,

$$e^x \geqslant 1 + x + \tfrac{1}{2}x^2$$

(b) Use part (a) to improve the estimate of $\int_0^1 e^{x^2}\,dx$ given in Exercise 103(b).

105. (a) Use mathematical induction to prove that for $x \geqslant 0$ and any positive integer n,

$$e^x \geqslant 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$

(b) Use part (a) to show that $e > 2.7$.

(c) Use part (a) to show that

$$\lim_{x \to \infty} \frac{e^x}{x^k} = \infty$$

for any positive integer k.