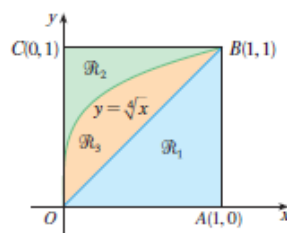## 5.2 Exercises

1–18 Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line. Sketch the region, the solid, and a typical disk or washer.

1. $y = 2 - \frac{1}{2}x$, $y = 0$, $x = 1$, $x = 2$; about the x-axis

2. $y = 1 - x^2$, $y = 0$; about the x-axis

3. $y = \sqrt{x - 1}$, $y = 0$, $x = 5$; about the x-axis

4. $y = \sqrt{25 - x^2}$, $y = 0$, $x = 2$, $x = 4$; about the x-axis

5. $x = 2\sqrt{y}$, $x = 0$, $y = 9$; about the y-axis

6. $x = y - y^2$, $x = 0$; about the y-axis

7. $y = x^3$, $y = x$, $x \geq 0$; about the x-axis

8. $y = \frac{1}{4}x^2$, $y = 5 - x^2$; about the x-axis

9. $y^2 = x$, $x = 2y$; about the y-axis

10. $y = \frac{1}{4}x^2$, $x = 2$, $y = 0$; about the y-axis

11. $y = x^2$, $x = y^2$; about $y = 1$

12. $y = x^2$, $y = 4$; about $y = 4$

13. $y = 1 + \sec x$, $y = 3$; about $y = 1$

14. $y = \sin x$, $y = \cos x$, $0 \leq x \leq \pi/4$; about $y = -1$

15. $y = x^3$, $y = 0$, $x = 1$; about $x = 2$

16. $y = x^2$, $x = y^2$; about $x = -1$

17. $x = y^2$, $x = 1 - y^2$; about $x = 3$

18. $y = x$, $y = 0$, $x = 2$, $x = 4$; about $x = 1$

19–30 Refer to the figure and find the volume generated by rotating the given region about the specified line.

19. $\mathcal{R}_1$ about OA

20. $\mathcal{R}_1$ about OC

21. $\mathcal{R}_1$ about AB

22. $\mathcal{R}_1$ about BC

23. $\mathcal{R}_2$ about OA

24. $\mathcal{R}_2$ about OC

25. $\mathcal{R}_2$ about AB

26. $\mathcal{R}_2$ about BC

27. $\mathcal{R}_3$ about OA

28. $\mathcal{R}_3$ about OC

29. $\mathcal{R}_3$ about AB

30. $\mathcal{R}_3$ about BC

31–34 Set up an integral for the volume of the solid obtained by rotating the region bounded by the given curves about the specified line. Then use your calculator to evaluate the integral correct to five decimal places.

31. $y = \tan x$, $y = 0$, $x = \pi/4$
   (a) About the x-axis   (b) About $y = -1$

32. $y = 0$, $y = \cos^2 x$, $-\pi/2 \le x \le \pi/2$
   (a) About the x-axis   (b) About $y = 1$

33. $x^2 + 4y^2 = 4$
   (a) About $y = 2$ (b) About $x = 2$

34. $y = x^2$, $x^2 + y^2 = 1$, $y \ge 0$
   (a) About the x-axis   (b) About the y-axis

35–36 Use a graph to find approximate x-coordinates of the points of intersection of the given curves. Then use your calculator to find (approximately) the volume of the solid obtained by rotating about the x-axis the region bounded by these curves.

35. $y = 2 + x^2 \cos x$, $y = x^4 + x + 1$

36. $y = x^4$, $y = 3x - x^3$

CAS **37–38** Use a computer algebra system to find the exact volume of the solid obtained by rotating the region bounded by the given curves about the specified line.

37. $y = \sin^2 x$, $y = 0$, $0 \le x \le \pi$;   about $y = -1$

38. $y = x^2 - 2x$, $y = x \cos(\pi x/4)$;   about $y = 2$

**39–42** Each integral represents the volume of a solid. Describe the solid.

39. $\pi \int_0^\pi \sin x \, dx$

40. $\pi \int_{-1}^1 (1 - y^2)^2 \, dy$

41. $\pi \int_0^1 (y^4 - y^8) \, dy$

42. $\pi \int_0^{\pi/2} [(1 + \cos x)^2 - 1^2] \, dx$

43. A CAT scan produces equally spaced cross-sectional views of a human organ that provide information about the organ otherwise obtained only by surgery. Suppose that a CAT scan of a human liver shows cross-sections spaced 1.5 cm apart. The liver is 15 cm long and the cross-sectional areas, in square centimeters, are 0, 18, 58, 79, 94, 106, 117, 128, 63, 39, and 0. Use the Midpoint Rule to estimate the volume of the liver.

44. A log 10 m long is cut at 1-meter intervals and its cross-sectional areas A (at a distance x from the end of the log) are listed in the table. Use the Midpoint Rule with $n = 5$ to estimate the volume of the log.

x (m)	A (m²)	x (m)	A (m²)
0	0.68	6	0.53
1	0.65	7	0.55
2	0.64	8	0.52
3	0.61	9	0.50
4	0.58	10	0.48
5	0.59		

45. (a) If the region shown in the figure is rotated about the x-axis to form a solid, use the Midpoint Rule with $n = 4$ to estimate the volume of the solid.
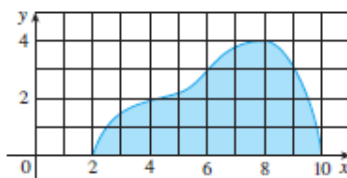
   (b) Estimate the volume if the region is rotated about the y-axis. Again use the Midpoint Rule with $n = 4$.

CAS **46.** (a) A model for the shape of a bird's egg is obtained by rotating about the x-axis the region under the graph of
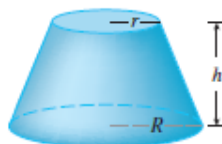
$$f(x) = (ax^3 + bx^2 + cx + d)\sqrt{1 - x^2}$$
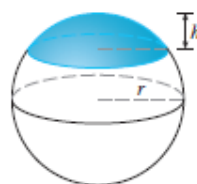
   Use a CAS to find the volume of such an egg.
   (b) For a Red-throated Loon, $a = -0.06$, $b = 0.04$, $c = 0.1$, and $d = 0.54$. Graph f and find the volume of an egg of this species.

47–59 Find the volume of the described solid S.

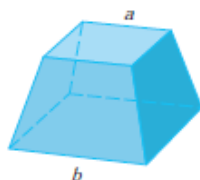**47.** A right circular cone with height h and base radius r

48. A frustum of a right circular cone with height h, lower base radius R, and top radius r

49. A cap of a sphere with radius r and cap height h

50. A frustum of a pyramid with square base of side b, square top of side a, and height h

What happens if $a = b$? What happens if $a = 0$?

51. A pyramid with height h and rectangular base with dimensions b and $2b$

52. A pyramid with height h and base an equilateral triangle with side a (a tetrahedron)

53. A tetrahedron with three mutually perpendicular faces and three mutually perpendicular edges with lengths 3 cm, 4 cm, and 5 cm

54. The base of S is a circular disk with radius r. Parallel cross-sections perpendicular to the base are squares.

55. The base of S is an elliptical region with boundary curve $9x^2 + 4y^2 = 36$. Cross-sections perpendicular to the x-axis are isosceles right triangles with hypotenuse in the base.

56. The base of S is the triangular region with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$. Cross-sections perpendicular to the y-axis are equilateral triangles.

57. The base of S is the same base as in Exercise 56, but cross-sections perpendicular to the x-axis are squares.

58. The base of S is the region enclosed by the parabola $y = 1 - x^2$ and the x-axis. Cross-sections perpendicular to the y-axis are squares.

59. The base of S is the same base as in Exercise 58, but cross-sections perpendicular to the x-axis are isosceles triangles with height equal to the base.

60. The base of S is a circular disk with radius r. Parallel cross-sections perpendicular to the base are isosceles triangles with height h and unequal side in the base.
 (a) Set up an integral for the volume of S.
 (b) By interpreting the integral as an area, find the volume of S.

61. (a) Set up an integral for the volume of a solid *torus* (the donut-shaped solid shown in the figure) with radii r and R.
 (b) By interpreting the integral as an area, find the volume of the torus.
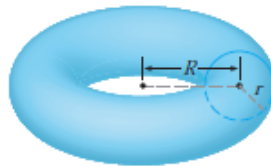
62. Solve Example 9 taking cross-sections to be parallel to the line of intersection of the two planes.

63. (a) Cavalieri's Principle states that if a family of parallel planes gives equal cross-sectional areas for two solids S_1 and S_2, then the volumes of S_1 and S_2 are equal. Prove this principle.
 (b) Use Cavalieri's Principle to find the volume of the oblique cylinder shown in the figure.
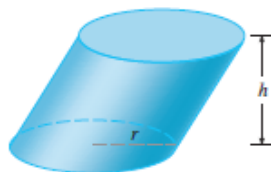
64. Find the volume common to two circular cylinders, each with radius r, if the axes of the cylinders intersect at right angles.
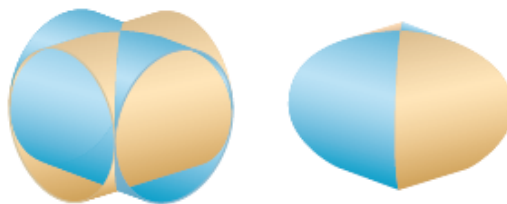
65. Find the volume common to two spheres, each with radius r, if the center of each sphere lies on the surface of the other sphere.

66. A bowl is shaped like a hemisphere with diameter 30 cm. A heavy ball with diameter 10 cm is placed in the bowl and water is poured into the bowl to a depth of h centimeters. Find the volume of water in the bowl.

67. A hole of radius r is bored through the middle of a cylinder of radius $R > r$ at right angles to the axis of the cylinder. Set up, but do not evaluate, an integral for the volume cut out.

68. A hole of radius r is bored through the center of a sphere of radius $R > r$. Find the volume of the remaining portion of the sphere.

69. Some of the pioneers of calculus, such as Kepler and Newton, were inspired by the problem of finding the volumes of wine

barrels. (In fact Kepler published a book *Stereometria doliorum* in 1615 devoted to methods for finding the volumes of barrels.) They often approximated the shape of the sides by parabolas.

(a) A barrel with height h and maximum radius R is constructed by rotating about the x-axis the parabola $y = R - cx^2$, $-h/2 \leqslant x \leqslant h/2$, where c is a positive constant. Show that the radius of each end of the barrel is $r = R - d$, where $d = ch^2/4$.

(b) Show that the volume enclosed by the barrel is

$$V = \tfrac{1}{3}\pi h\left(2R^2 + r^2 - \tfrac{2}{5}d^2\right)$$

70. Suppose that a region $\mathcal{R}$ has area A and lies above the x-axis. When $\mathcal{R}$ is rotated about the x-axis, it sweeps out a solid with volume V_1. When $\mathcal{R}$ is rotated about the line $y = -k$ (where k is a positive number), it sweeps out a solid with volume V_2. Express V_2 in terms of V_1, k, and A.