EXERCISES 4.9

Answers to selected odd-numbered problems begin on page ANS-6.

In Problems 1–20 solve the given system of differential equations by systematic elimination.

1. $\dfrac{dx}{dt} = 2x - y$, $\dfrac{dy}{dt} = x$

2. $\dfrac{dx}{dt} = 4x + 7y$, $\dfrac{dy}{dt} = x - 2y$

3. $\dfrac{dx}{dt} = -y + t$, $\dfrac{dy}{dt} = x - t$

4. $\dfrac{dx}{dt} - 4y = 1$, $\dfrac{dy}{dt} + x = 2$

5. $(D^2 + 5)x - 2y = 0$
$-2x + (D^2 + 2)y = 0$

6. $(D + 1)x + (D - 1)y = 2$
$3x + (D + 2)y = -1$

7. $\dfrac{d^2x}{dt^2} = 4y + e^t$, $\dfrac{d^2y}{dt^2} = 4x - e^t$

8. $\dfrac{d^2x}{dt^2} + \dfrac{dy}{dt} = -5x$, $\dfrac{dx}{dt} + \dfrac{dy}{dt} = -x + 4y$

9. $Dx + D^2y = e^{3t}$
$(D + 1)x + (D - 1)y = 4e^{3t}$

10. $D^2x - Dy = t$
$(D + 3)x + (D + 3)y = 2$

11. $(D^2 - 1)x - y = 0$
$(D - 1)x + Dy = 0$

12. $(2D^2 - D - 1)x - (2D + 1)y = 1$
$(D - 1)x + Dy = -1$

13. $2\dfrac{dx}{dt} - 5x + \dfrac{dy}{dt} = e^t$
$\dfrac{dx}{dt} - x + \dfrac{dy}{dt} = 5e^t$

14. $\dfrac{dx}{dt} + \dfrac{dy}{dt} = e^t$
$-\dfrac{d^2x}{dt^2} + \dfrac{dx}{dt} + x + y = 0$

15. $(D - 1)x + (D^2 + 1)y = 1$
$(D^2 - 1)x + (D + 1)y = 2$

16. $D^2x - 2(D^2 + D)y = \sin t$
$x + Dy = 0$

17. $Dx = y$
$Dy = z$
$Dz = x$

18. $Dx + z = e^t$
$(D - 1)x + Dy + Dz = 0$
$x + 2y + Dz = e^t$

19. $\dfrac{dx}{dt} = 6y$, $\dfrac{dy}{dt} = x + z$, $\dfrac{dz}{dt} = x + y$

20. $\dfrac{dx}{dt} = -x + z$, $\dfrac{dy}{dt} = -y + z$, $\dfrac{dz}{dt} = -x + y$

In Problems 21 and 22 solve the given initial-value problem.

21. $\dfrac{dx}{dt} = -5x - y$, $\dfrac{dy}{dt} = 4x - y$
$x(1) = 0, y(1) = 1$

22. $\dfrac{dx}{dt} = y - 1$, $\dfrac{dy}{dt} = -3x + 2y$
$x(0) = 0, y(0) = 0$

Mathematical Models

23. **Projectile Motion** A projectile shot from a gun has weight $w = mg$ and velocity $\mathbf{v}$ tangent to its path of motion. Ignoring air resistance and all other forces acting on the projectile except its weight, determine a system of differential equations that describes its path of motion. See Figure 4.9.2. Solve the system. [*Hint*: Use Newton's second law of motion in the x and y directions.]

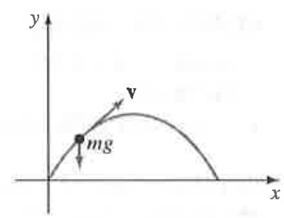

FIGURE 4.9.2 Path of projectile in Problem 23

24. **Projectile Motion with Air Resistance** Determine a system of differential equations that describes the path of motion in Problem 23 if air resistance is a retarding force $\mathbf{k}$ (of magnitude k) acting tangent to the path of the projectile but opposite to its motion. See Figure 4.9.3. Solve the system. [*Hint*: $\mathbf{k}$ is a multiple of velocity, say, $\beta\mathbf{v}$.]

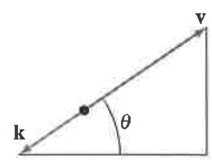

FIGURE 4.9.3 Forces in Problem 24

Discussion Problems

25. Examine and discuss the following system:

$$Dx - 2Dy = t^2$$
$$(D + 1)x - 2(D + 1)y = 1.$$

Computer Lab Assignments

26. Reexamine Figure 4.9.1 in Example 3. Then use a root-finding application to determine when tank B contains more salt than tank A.

27. **(a)** Reread Problem 8 of Exercises 3.3. In that problem you were asked to show that the system of differential equations

$$\frac{dx_1}{dt} = -\frac{1}{50}x_1$$
$$\frac{dx_2}{dt} = \frac{1}{50}x_1 - \frac{2}{75}x_2$$
$$\frac{dx_3}{dt} = \frac{2}{75}x_2 - \frac{1}{25}x_3$$

is a model for the amounts of salt in the connected mixing tanks A, B, and C shown in Figure 3.3.7. Solve the system subject to $x_1(0) = 15$, $x_2(t) = 10$, $x_3(t) = 5$.

(b) Use a CAS to graph $x_1(t)$, $x_2(t)$, and $x_3(t)$ in the same coordinate plane (as in Figure 4.9.1) on the interval [0, 200].

(c) Because only pure water is pumped into Tank A, it stands to reason that the salt will eventually be flushed out of all three tanks. Use a root-findin application of a CAS to determine the time when the amount of salt in each tank is less than or equal to 0.5 pound. When will the amounts of salt $x_1(t)$, $x_2(t)$, and $x_3(t)$ be simultaneously less than or equal to 0.5 pound?

4.10 NONLINEAR DIFFERENTIAL EQUATIONS

REVIEW MATERIAL

- Sections 2.2 and 2.5
- Section 4.2
- A review of Taylor series from calculus is also recommended.

INTRODUCTION The difficulties that surround higher-order *nonlinear* differential equations and the few methods that yield analytic solutions are examined next. Two of the solution methods considered in this section employ a change of variable to reduce a nonlinear second-order DE to a first-order DE. In that sense these methods are analogous to the material in Section 4.2

Some Differences There are several significan differences between linear and nonlinear differential equations. We saw in Section 4.1 that homogeneous linear equations of order two or higher have the property that a linear combination of solutions is also a solution (Theorem 4.1.2). Nonlinear equations do not possess this property of superposability. See Problems 1 and 18 in Exercises 4.10. We can fin general solutions of linear first-orde DEs and higher-order equations with constant coefficients Even when we can solve a nonlinear first-orde differential equation in the form of a one-parameter family, this family does not, as a rule, represent a general solution. Stated another way, nonlinear first-orde DEs can possess singular solutions, whereas linear equations cannot. But the major difference between linear and nonlinear equations of order two or higher lies in the realm of solvability. Given a linear equation, there is a chance that we can fin some form of a solution that we can look at—an explicit solution or perhaps a solution in the form of an infinit series (see Chapter 6). On the other hand, nonlinear higher-order differential equations virtually defy solution by analytical methods. Although this might sound disheartening, there are still things that can be done. As was pointed out at the end of Section 1.3, we can always analyze a nonlinear DE qualitatively and numerically.

Let us make it clear at the outset that nonlinear higher-order differential equations are important—dare we say even more important than linear equations?—because as we fine-tune the mathematical model of, say, a physical system, we also increase the likelihood that this higher-resolution model will be nonlinear.

We begin by illustrating an analytical method that *occasionally* enables us to find explicit/implicit solutions of special kinds of nonlinear second-order differential equations.