See Problems 37 and 38 in Exercises 4.7.

A Different Form A second-order equation of the form

$$a(x - x_0)^2 \frac{d^2y}{dx^2} + b(x - x_0)\frac{dy}{dx} + cy = 0 \tag{6}$$

is also a Cauchy-Euler equation. Observe that (6) reduces to (1) when $x_0 = 0$.

We can solve (6) as we did (1), namely, seeking solutions of $y = (x - x_0)^m$ and using

$$\frac{dy}{dx} = m(x - x_0)^{m-1} \quad \text{and} \quad \frac{d^2y}{dx^2} = m(m - 1)(x - x_0)^{m-2}.$$

Alternatively, we can reduce (6) to the familiar form (1) by means of the change of independent variable $t = x - x_0$, solving the reduced equation, and resubstituting. See Problems 39–42 in Exercises 4.7.

EXERCISES 4.7

Answers to selected odd-numbered problems begin on page ANS-6.

In Problems 1–18 solve the given differential equation.

1. $x^2y'' - 2y = 0$
2. $4x^2y'' + y = 0$
3. $xy'' + y' = 0$
4. $xy'' - 3y' = 0$
5. $x^2y'' + xy' + 4y = 0$
6. $x^2y'' + 5xy' + 3y = 0$
7. $x^2y'' - 3xy' - 2y = 0$
8. $x^2y'' + 3xy' - 4y = 0$
9. $25x^2y'' + 25xy' + y = 0$
10. $4x^2y'' + 4xy' - y = 0$
11. $x^2y'' + 5xy' + 4y = 0$
12. $x^2y'' + 8xy' + 6y = 0$
13. $3x^2y'' + 6xy' + y = 0$
14. $x^2y'' - 7xy' + 41y = 0$
15. $x^3y''' - 6y = 0$
16. $x^3y''' + xy' - y = 0$
17. $xy^{(4)} + 6y''' = 0$
18. $x^4y^{(4)} + 6x^3y''' + 9x^2y'' + 3xy' + y = 0$

In Problems 19–24 solve the given differential equation by variation of parameters.

19. $xy'' - 4y' = x^4$
20. $2x^2y'' + 5xy' + y = x^2 - x$
21. $x^2y'' - xy' + y = 2x$
22. $x^2y'' - 2xy' + 2y = x^4e^x$
23. $x^2y'' + xy' - y = \ln x$
24. $x^2y'' + xy' - y = \dfrac{1}{x + 1}$

In Problems 25–30 solve the given initial-value problem. Use a graphing utility to graph the solution curve.

25. $x^2y'' + 3xy' = 0, \quad y(1) = 0, y'(1) = 4$
26. $x^2y'' - 5xy' + 8y = 0, \quad y(2) = 32, y'(2) = 0$
27. $x^2y'' + xy' + y = 0, \quad y(1) = 1, y'(1) = 2$
28. $x^2y'' - 3xy' + 4y = 0, \quad y(1) = 5, y'(1) = 3$
29. $xy'' + y' = x, \quad y(1) = 1, y'(1) = -\frac{1}{2}$
30. $x^2y'' - 5xy' + 8y = 8x^6, \quad y(\frac{1}{2}) = 0, y'(\frac{1}{2}) = 0$

In Problems 31–36 use the substitution $x = e^t$ to transform the given Cauchy-Euler equation to a differential equation with constant coefficients. Solve the original equation by solving the new equation using the procedures in Sections 4.3–4.5.

31. $x^2y'' + 9xy' - 20y = 0$
32. $x^2y'' - 9xy' + 25y = 0$
33. $x^2y'' + 10xy' + 8y = x^2$
34. $x^2y'' - 4xy' + 6y = \ln x^2$
35. $x^2y'' - 3xy' + 13y = 4 + 3x$
36. $x^3y''' - 3x^2y'' + 6xy' - 6y = 3 + \ln x^3$

In Problems 37 and 38 use the substitution $t = -x$ to solve the given initial-value problem on the interval $(-\infty, 0)$.

37. $4x^2y'' + y = 0, \quad y(-1) = 2, y'(-1) = 4$
38. $x^2y'' - 4xy' + 6y = 0, \quad y(-2) = 8, y'(-2) = 0$

In Problems 39 and 40 use $y = (x - x_0)^m$ to solve the given differential equation.

39. $(x + 3)^2y'' - 8(x - 1)y' + 14y = 0$
40. $(x - 1)^2y'' - (x - 1)y' + 5y = 0$

In Problems 41 and 42 use the substitution $t = x - x_0$ to solve the given differential equation.

41. $(x + 2)^2y'' + (x + 2)y' + y = 0$

42. $(x - 4)^2y'' - 5(x - 4)y' + 9y = 0$

Discussion Problems

43. Give the largest interval over which the general solution of Problem 42 is defined

44. Can a Cauchy-Euler differential equation of lowest order with real coefficients be found if it is known that 2 and $1 - i$ are roots of its auxiliary equation? Carry out your ideas.

45. The initial-conditions $y(0) = y_0$, $y'(0) = y_1$ apply to each of the following differential equations:

$$x^2y'' = 0,$$

$$x^2y'' - 2xy' + 2y = 0,$$

$$x^2y'' - 4xy' + 6y = 0.$$

For what values of y_0 and y_1 does each initial-value problem have a solution?

46. What are the x-intercepts of the solution curve shown in Figure 4.7.1? How many x-intercepts are there for $0 < x < \frac{1}{2}$?

Computer Lab Assignments

In Problems 47–50 solve the given differential equation by using a CAS to find the (approximate) roots of the auxiliary equation.

47. $2x^3y''' - 10.98x^2y'' + 8.5xy' + 1.3y = 0$

48. $x^3y''' + 4x^2y'' + 5xy' - 9y = 0$

49. $x^4y^{(4)} + 6x^3y''' + 3x^2y'' - 3xy' + 4y = 0$

50. $x^4y^{(4)} - 6x^3y''' + 33x^2y'' - 105xy' + 169y = 0$

51. Solve $x^3y''' - x^2y'' - 2xy' + 6y = x^2$ by variation of parameters. Use a CAS as an aid in computing roots of the auxiliary equation and the determinants given in (15) of Section 4.6.

4.8 GREEN'S FUNCTIONS

REVIEW MATERIAL

- See the *Remarks* at the end of Section 4.1 for the definitions of *response*, *input*, and *output*.
- Differential operators in Section 4.1 and Section 4.5
- The method of variation of parameters in Section 4.6

INTRODUCTION We will see in Chapter 5 that the linear second-order differential equation

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \tag{1}$$

plays an important role in many applications. In the mathematical analysis of physical systems it is often desirable to express the **response** or **output** $y(x)$ of (1) subject to either initial conditions or boundary conditions directly in terms of the **forcing function** or **input** $g(x)$. In this manner the response of the system can quickly be analyzed for different forcing functions.

To see how this is done, we start by examining solutions of initial-value problems in which the DE (1) has been put into the standard form

$$y'' + P(x)y' + Q(x)y = f(x) \tag{2}$$

by dividing the equation by the lead coefficient $a_2(x)$. We also assume throughout this section that the coefficient functions $P(x)$, $Q(x)$, and $f(x)$ are continuous on some common interval I.

4.8.1 INITIAL-VALUE PROBLEMS

Three Initial-Value Problems We will see as the discussion unfolds that the solution $y(x)$ of the second order initial-value problem

$$y'' + P(x)y' + Q(x)y = f(x), \quad y(x_0) = y_0, \quad y'(x_0) = y_1 \tag{3}$$