where the u'_k, $k = 1, 2, \ldots, n$ are determined by the n equations

$$\begin{aligned} y_1u'_1 + \quad y_2u'_2 + \cdots + \quad y_nu'_n &= 0 \\ y'_1u'_1 + \quad y'_2u'_2 + \cdots + \quad y'_nu'_n &= 0 \\ &\vdots \\ y_1^{(n-1)}u'_1 + y_2^{(n-1)}u'_2 + \cdots + y_n^{(n-1)}u'_n &= f(x). \end{aligned} \tag{14}$$

The first $n - 1$ equations in this system, like $y_1u'_1 + y_2u'_2 = 0$ in (8), are assumptions that are made to simplify the resulting equation after $y_p = u_1(x)y_1(x) + \cdots + u_n(x)y_n(x)$ is substituted in (13). In this case Cramer's Rule gives

$$u'_k = \frac{W_k}{W}, \quad k = 1, 2, \ldots, n,$$

where W is the Wronskian of $y_1, y_2, \ldots, y_n$ and W_k is the determinant obtained by replacing the kth column of the Wronskian by the column consisting of the right-hand side of (14)—that is, the column consisting of $(0, 0, \ldots, f(x))$. When $n = 2$, we get (9). When $n = 3$, the particular solution is $y_p = u_1y_1 + u_2y_2 + u_3y_3$, where y_1, y_2, and y_3 constitute a linearly independent set of solutions of the associated homogeneous DE and u_1, u_2, u_3 are determined from

$$u'_1 = \frac{W_1}{W}, \quad u'_2 = \frac{W_2}{W}, \quad u'_3 = \frac{W_3}{W}, \tag{15}$$

$$W_1 = \begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y'_2 & y'_3 \\ f(x) & y''_2 & y''_3 \end{vmatrix}, \quad W_2 = \begin{vmatrix} y_1 & 0 & y_3 \\ y'_1 & 0 & y'_3 \\ y''_1 & f(x) & y''_3 \end{vmatrix}, \quad W_3 = \begin{vmatrix} y_1 & y_2 & 0 \\ y'_1 & y'_2 & 0 \\ y''_1 & y''_2 & f(x) \end{vmatrix}, \quad \text{and} \quad W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y'_1 & y'_2 & y'_3 \\ y''_1 & y''_2 & y''_3 \end{vmatrix}.$$

See Problems 25–28 in Exercises 4.6.

REMARKS

(i) Variation of parameters has a distinct advantage over the method of undetermined coefficients in that it will *always* yield a particular solution y_p provided that the associated homogeneous equation can be solved. The present method is not limited to a function $f(x)$ that is a combination of the four types listed on page 140. As we shall see in the next section, variation of parameters, unlike undetermined coefficients, is applicable to linear DEs with variable coefficients

(ii) In the problems that follow, do not hesitate to simplify the form of y_p. Depending on how the antiderivatives of u'_1 and u'_2 are found, you might not obtain the same y_p as given in the answer section. For example, in Problem 3 in Exercises 4.6 both $y_p = \frac{1}{2}\sin x - \frac{1}{2}x\cos x$ and $y_p = \frac{1}{4}\sin x - \frac{1}{2}x\cos x$ are valid answers. In either case the general solution $y = y_c + y_p$ simplifies to $y = c_1\cos x + c_2\sin x - \frac{1}{2}x\cos x$. Why?

EXERCISES 4.6

Answers to selected odd-numbered problems begin on page ANS-6.

In Problems 1–18 solve each differential equation by variation of parameters.

1. $y'' + y = \sec x$
2. $y'' + y = \tan x$
3. $y'' + y = \sin x$
4. $y'' + y = \sec\theta\tan\theta$
5. $y'' + y = \cos^2 x$
6. $y'' + y = \sec^2 x$
7. $y'' - y = \cosh x$
8. $y'' - y = \sinh 2x$
9. $y'' - 4y = \dfrac{e^{2x}}{x}$
10. $y'' - 9y = \dfrac{9x}{e^{3x}}$

11. $y'' + 3y' + 2y = \dfrac{1}{1 + e^x}$

12. $y'' - 2y' + y = \dfrac{e^x}{1 + x^2}$

13. $y'' + 3y' + 2y = \sin e^x$

14. $y'' - 2y' + y = e^t \arctan t$

15. $y'' + 2y' + y = e^{-t} \ln t$

16. $2y'' + 2y' + y = 4\sqrt{x}$

17. $3y'' - 6y' + 6y = e^x \sec x$

18. $4y'' - 4y' + y = e^{x/2}\sqrt{1 - x^2}$

In Problems 19–22 solve each differential equation by variation of parameters, subject to the initial conditions $y(0) = 1, y'(0) = 0$.

19. $4y'' - y = xe^{x/2}$

20. $2y'' + y' - y = x + 1$

21. $y'' + 2y' - 8y = 2e^{-2x} - e^{-x}$

22. $y'' - 4y' + 4y = (12x^2 - 6x)e^{2x}$

In Problems 23 and 24 the indicated functions are known linearly independent solutions of the associated homogeneous differential equation on (0,). Find the general solution of the given nonhomogeneous equation.

23. $x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = x^{3/2}$;
$y_1 = x^{-1/2} \cos x,\ y_2 = x^{-1/2} \sin x$

24. $x^2y'' + xy' + y = \sec(\ln x)$;
$y_1 = \cos(\ln x),\ y_2 = \sin(\ln x)$

In Problems 25–28 solve the given third-order differential equation by variation of parameters.

25. $y''' + y' = \tan x$

26. $y''' + 4y' = \sec 2x$

27. $y''' - 2y'' - y' + 2y = e^{4x}$

28. $y''' - 3y'' + 2y' = \dfrac{e^{2x}}{1 + e^x}$

Discussion Problems

In Problems 29 and 30 discuss how the methods of undetermined coefficients and variation of parameters can be combined to solve the given differential equation. Carry out your ideas.

29. $3y'' - 6y' + 30y = 15 \sin x + e^x \tan 3x$

30. $y'' - 2y' + y = 4x^2 - 3 + x^{-1}e^x$

31. What are the intervals of definition of the general solutions in Problems 1, 7, 9, and 18? Discuss why the interval of definition of the general solution in Problem 24 is *not* (0,).

32. Find the general solution of $x^4y'' + x^3y' - 4x^2y = 1$ given that $y_1 = x^2$ is a solution of the associated homogeneous equation.

4.7 CAUCHY-EULER EQUATION

REVIEW MATERIAL

- Review the concept of the auxiliary equation in Section 4.3.

INTRODUCTION The same relative ease with which we were able to find explicit solutions of higher-order linear differential equations with constant coefficients in the preceding sections does not, in general, carry over to linear equations with variable coefficients. We shall see in Chapter 6 that when a linear DE has variable coefficients, the best that we can *usually* expect is to find a solution in the form of an infinite series. However, the type of differential equation that we consider in this section is an exception to this rule; it is a linear equation with variable coefficients whose general solution can always be expressed in terms of powers of x, sines, cosines, and logarithmic functions. Moreover, its method of solution is quite similar to that for constant-coefficient equations in that an auxiliary equation must be solved.

Cauchy-Euler Equation A linear differential equation of the form

$$a_nx^n\frac{d^ny}{dx^n} + a_{n-1}x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1x\frac{dy}{dx} + a_0y = g(x),$$