algebra systems are also able, by means of their *dsolve* commands, to provide explicit solutions of homogeneous linear constant-coefficient di ferential equations.

In the classic text *Differential Equations* by Ralph Palmer Agnew* (used by the author as a student) the following statement is made:

It is not reasonable to expect students in this course to have computing skill and equipment necessary for efficient solving of equations such a

$$4.317\frac{d^4y}{dx^4} + 2.179\frac{d^3y}{dx^3} + 1.416\frac{d^2y}{dx^2} + 1.295\frac{dy}{dx} + 3.169y = 0. \qquad (13)$$

Although it is debatable whether computing skills have improved in the intervening years, it is a certainty that technology has. If one has access to a computer algebra system, equation (13) could now be considered reasonable. After simplification and some relabeling of output, *Mathematica* yields the (approximate) general solution

$$y = c_1e^{-0.728852x}\cos(0.618605x) + c_2e^{-0.728852x}\sin(0.618605x)$$
$$+ c_3e^{0.476478x}\cos(0.759081x) + c_4e^{0.476478x}\sin(0.759081x).$$

Finally, if we are faced with an initial-value problem consisting of, say, a fourth-order equation, then to fit the general solution of the DE to the four initial conditions, we must solve four linear equations in four unknowns (the c_1, c_2, c_3, c_4 in the general solution). Using a CAS to solve the system can save lots of time. See Problems 69 and 70 in Exercises 4.3 and Problem 41 in Chapter 4 in Review.

*McGraw-Hill, New York, 1960.

EXERCISES 4.3

Answers to selected odd-numbered problems begin on page ANS-5.

In Problems 1–14 find the general solution of the given second-order differential equation.

1. $4y'' + y' = 0$
2. $y'' - 36y = 0$
3. $y'' - y' - 6y = 0$
4. $y'' - 3y' + 2y = 0$
5. $y'' + 8y' + 16y = 0$
6. $y'' - 10y' + 25y = 0$
7. $12y'' - 5y' - 2y = 0$
8. $y'' + 4y' - y = 0$
9. $y'' + 9y = 0$
10. $3y'' + y = 0$
11. $y'' - 4y' + 5y = 0$
12. $2y'' + 2y' + y = 0$
13. $3y'' + 2y' + y = 0$
14. $2y'' - 3y' + 4y = 0$

In Problems 15–28 find the general solution of the given higher-order differential equation.

15. $y''' - 4y'' - 5y' = 0$
16. $y''' - y = 0$
17. $y''' - 5y'' + 3y' + 9y = 0$
18. $y''' + 3y'' - 4y' - 12y = 0$
19. $\frac{d^3u}{dt^3} + \frac{d^2u}{dt^2} - 2u = 0$
20. $\frac{d^3x}{dt^3} - \frac{d^2x}{dt^2} - 4x = 0$
21. $y''' + 3y'' + 3y' + y = 0$
22. $y''' - 6y'' + 12y' - 8y = 0$
23. $y^{(4)} + y''' + y'' = 0$
24. $y^{(4)} - 2y'' + y = 0$
25. $16\frac{d^4y}{dx^4} + 24\frac{d^2y}{dx^2} + 9y = 0$
26. $\frac{d^4y}{dx^4} - 7\frac{d^2y}{dx^2} - 18y = 0$
27. $\frac{d^5u}{dr^5} + 5\frac{d^4u}{dr^4} - 2\frac{d^3u}{dr^3} - 10\frac{d^2u}{dr^2} + \frac{du}{dr} + 5u = 0$
28. $2\frac{d^5x}{ds^5} - 7\frac{d^4x}{ds^4} + 12\frac{d^3x}{ds^3} + 8\frac{d^2x}{ds^2} = 0$

In Problems 29–36 solve the given initial-value problem.

29. $y'' + 16y = 0, \quad y(0) = 2, y'(0) = -2$
30. $\frac{d^2y}{d\theta^2} + y = 0, \quad y(\pi/3) = 0, y'(\pi/3) = 2$

31. $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} - 5y = 0, \quad y(1) = 0, y'(1) = 2$

32. $4y'' - 4y' - 3y = 0, \quad y(0) = 1, y'(0) = 5$

33. $y'' + y' + 2y = 0, \quad y(0) = y'(0) = 0$

34. $y'' - 2y' + y = 0, \quad y(0) = 5, y'(0) = 10$

35. $y''' + 12y'' + 36y' = 0, \quad y(0) = 0, y'(0) = 1, y''(0) = -7$

36. $y''' + 2y'' - 5y' - 6y = 0, \quad y(0) = y'(0) = 0, \ y''(0) = 1$

In Problems 37–40 solve the given boundary-value problem.

37. $y'' - 10y' + 25y = 0, \quad y(0) = 1, y(1) = 0$

38. $y'' + 4y = 0, \quad y(0) = 0, y(\pi) = 0$

39. $y'' + y = 0, \quad y'(0) = 0, y'(\pi/2) = 0$

40. $y'' - 2y' + 2y = 0, \quad y(0) = 1, y(\pi) = 1$

In Problems 41 and 42 solve the given problem first using the form of the general solution given in (10). Solve again, this time using the form given in (11).

41. $y'' - 3y = 0, \quad y(0) = 1, y'(0) = 5$

42. $y'' - y = 0, \quad y(0) = 1, y'(1) = 0$

In Problems 43–48 each figure represents the graph of a particular solution of one of the following differential equations:

(a) $y'' - 3y' - 4y = 0$ (b) $y'' + 4y = 0$
(c) $y'' + 2y' + y = 0$ (d) $y'' + y = 0$
(e) $y'' + 2y' + 2y = 0$ (f) $y'' - 3y' + 2y = 0$

Match a solution curve with one of the differential equations. Explain your reasoning.

43.

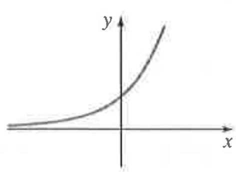

FIGURE 4.3.2 Graph for Problem 43

44.

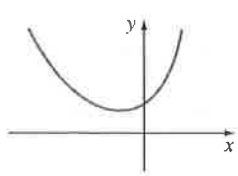

FIGURE 4.3.3 Graph for Problem 44

45.

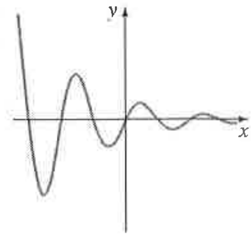

FIGURE 4.3.4 Graph for Problem 45

46.

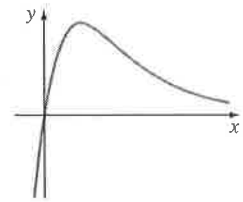

FIGURE 4.3.5 Graph for Problem 46

47.

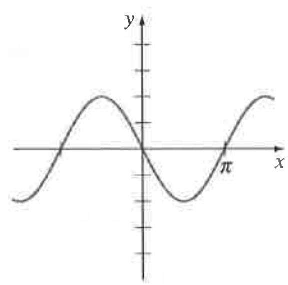

FIGURE 4.3.6 Graph for Problem 47

48.

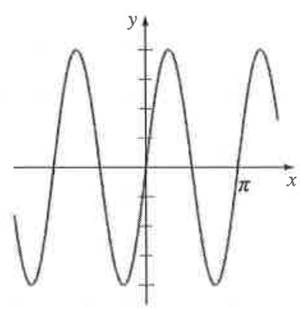

FIGURE 4.3.7 Graph for Problem 48

In Problems 49–58 find a homogeneous linear differential equation with constant coefficients whose general solution is given.

49. $y = c_1e^x + c_2e^{5x}$

50. $y = c_1e^{-4x} + c_2e^{-3x}$

51. $y = c_1 + c_2e^{2x}$

52. $y = c_1e^{10x} + c_2xe^{10x}$

53. $y = c_1 \cos 3x + c_2 \sin 3x$

54. $y = c_1 \cosh 7x + c_2 \sinh 7x$

55. $y = c_1e^{-x}\cos x + c_2e^{-x}\sin x$

56. $y = c_1 + c_2e^{2x}\cos 5x + c_3e^{2x}\sin 5x$

57. $y = c_1 + c_2x + c_3e^{8x}$

58. $y = c_1 \cos x + c_2 \sin x + c_3 \cos 2x + c_4 \sin 2x$

Discussion Problems

59. Two roots of a cubic auxiliary equation with real coeffi cients are $m_1 = -\frac{1}{2}$ and $m_2 = 3 + i$. What is the corresponding homogeneous linear differential equation? Discuss: Is your answer unique?

60. Find the general solution of $2y''' + 7y'' + 4y' - 4y = 0$ if $m_1 = \frac{1}{2}$ is one root of its auxiliary equation.

61. Find the general solution of $y''' + 6y'' + y' - 34y = 0$ if it is known that $y_1 = e^{-4x}\cos x$ is one solution.

62. To solve $y^{(4)} + y = 0$, we must find the roots of $m^4 + 1 = 0$. This is a trivial problem using a CAS but can also be done by hand working with complex numbers. Observe that $m^4 + 1 = (m^2 + 1)^2 - 2m^2$. How does this help? Solve the differential equation.

63. Verify that $y = \sinh x - 2\cos(x + \pi/6)$ is a particular solution of $y^{(4)} - y = 0$. Reconcile this particular solution with the general solution of the DE.

64. Consider the boundary-value problem $y'' + \lambda y = 0$, $y(0) = 0$, $y(\pi/2) = 0$. Discuss: Is it possible to determine values of λ so that the problem possesses **(a)** trivial solutions? **(b)** nontrivial solutions?

Computer Lab Assignments

In Problems 65–68 use a computer either as an aid in solving the auxiliary equation or as a means of directly obtaining the general solution of the given differential equation. If you use a CAS to obtain the general solution, simplify the output and, if necessary, write the solution in terms of real functions.

65. $y''' - 6y'' + 2y' + y = 0$

66. $6.11y''' + 8.59y'' + 7.93y' + 0.778y = 0$

67. $3.15y^{(4)} - 5.34y'' + 6.33y' - 2.03y = 0$

68. $y^{(4)} + 2y'' - y' + 2y = 0$

In Problems 69 and 70 use a CAS as an aid in solving the auxiliary equation. Form the general solution of the differential equation. Then use a CAS as an aid in solving the system of equations for the coefficients c_i, $i = 1, 2, 3, 4$ that results when the initial conditions are applied to the general solution.

69. $2y^{(4)} + 3y''' - 16y'' + 15y' - 4y = 0$,
$y(0) = -2, y'(0) = 6, y''(0) = 3, y'''(0) = \frac{1}{2}$

70. $y^{(4)} - 3y''' + 3y'' - y' = 0$,
$y(0) = y'(0) = 0, y''(0) = y'''(0) = 1$

4.4 UNDETERMINED COEFFICIENTS—SUPERPOSITION APPROACH*

REVIEW MATERIAL

- Review Theorems 4.1.6 and 4.1.7 (Section 4.1)

INTRODUCTION To solve a nonhomogeneous linear differential equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = g(x), \tag{1}$$

we must do two things:

- find the complementary function y_c and
- find *any* particular solution y_p of the nonhomogeneous equation (1).

Then, as was discussed in Section 4.1, the general solution of (1) is $y = y_c + y_p$. The complementary function y_c is the general solution of the associated homogeneous DE of (1), that is,

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0.$$

In Section 4.3 we saw how to solve these kinds of equations when the coefficients were constants. Our goal in the present section is to develop a method for obtaining particular solutions.

*Note to the Instructor: In this section the method of undetermined coefficients is developed from th viewpoint of the superposition principle for nonhomogeneous equations (Theorem 4.7.1). In Section 4.5 an entirely different approach will be presented, one utilizing the concept of differential annihilator operators. Take your pick.