By choosing $c_1 = 1$ and $c_2 = 0$, we find from $y = u(x)y_1(x)$ that a second solution of equation (3) is

$$y_2 = y_1(x) \int \frac{e^{-\int P(x)\,dx}}{y_1^2(x)}\,dx. \tag{5}$$

It makes a good review of differentiation to verify that the function $y_2(x)$ defined in (5) satisfies equation (3) and that y_1 and y_2 are linearly independent on any interval on which $y_1(x)$ is not zero.

EXAMPLE 2 A Second Solution by Formula (5)

The function $y_1 = x^2$ is a solution of $x^2y'' - 3xy' + 4y = 0$. Find the general solution of the differential equation on the interval (0,).

SOLUTION From the standard form of the equation,

$$y'' - \frac{3}{x}y' + \frac{4}{x^2}y = 0,$$

we find from (5

$$y_2 = x^2 \int \frac{e^{3\int dx/x}}{x^4}\,dx \quad \leftarrow e^{3\int dx/x} = e^{\ln x^3} = x^3$$

$$= x^2 \int \frac{dx}{x} = x^2 \ln x.$$

The general solution on the interval (0,) is given by $y = c_1y_1 + c_2y_2$; that is, $y = c_1x^2 + c_2x^2 \ln x$. ≡

REMARKS

(*i*) The derivation and use of formula (5) have been illustrated here because this formula appears again in the next section and in Sections 4.7 and 6.3. We use (5) simply to save time in obtaining a desired result. Your instructor will tell you whether you should memorize (5) or whether you should know the first principles of reduction of order.

(*ii*) Reduction of order can be used to find the general solution of a nonhomogeneous equation $a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$ whenever a solution y_1 of the associated homogeneous equation is known. See Problems 17–20 in Exercises 4.2.

EXERCISES 4.2

Answers to selected odd-numbered problems begin on page ANS-5.

In Problems 1–16 the indicated function $y_1(x)$ is a solution of the given differential equation. Use reduction of order or formula (5), as instructed, to find a second solution $y_2(x)$.

1. $y'' - 4y' + 4y = 0; \quad y_1 = e^{2x}$
2. $y'' + 2y' + y = 0; \quad y_1 = xe^{-x}$
3. $y'' + 16y = 0; \quad y_1 = \cos 4x$
4. $y'' + 9y = 0; \quad y_1 = \sin 3x$
5. $y'' - y = 0; \quad y_1 = \cosh x$
6. $y'' - 25y = 0; \quad y_1 = e^{5x}$
7. $9y'' - 12y' + 4y = 0; \quad y_1 = e^{2x/3}$
8. $6y'' + y' - y = 0; \quad y_1 = e^{x/3}$
9. $x^2y'' - 7xy' + 16y = 0; \quad y_1 = x^4$
10. $x^2y'' + 2xy' - 6y = 0; \quad y_1 = x^2$
11. $xy'' + y' = 0; \quad y_1 = \ln x$
12. $4x^2y'' + y = 0; \quad y_1 = x^{1/2} \ln x$
13. $x^2y'' - xy' + 2y = 0; \quad y_1 = x \sin(\ln x)$
14. $x^2y'' - 3xy' + 5y = 0; \quad y_1 = x^2 \cos(\ln x)$

15. $(1-2x-x^2)y''+2(1+x)y'-2y=0;\quad y_1=x+1$

16. $(1-x^2)y''+2xy'=0;\quad y_1=1$

In Problems 17–20 the indicated function $y_1(x)$ is a solution of the associated homogeneous equation. Use the method of reduction of order to find a second solution $y_2(x)$ of the homogeneous equation and a particular solution of the given nonhomogeneous equation.

17. $y''-4y=2;\quad y_1=e^{-2x}$

18. $y''+y'=1;\quad y_1=1$

19. $y''-3y'+2y=5e^{3x};\quad y_1=e^x$

20. $y''-4y'+3y=x;\quad y_1=e^x$

Discussion Problems

21. **(a)** Give a convincing demonstration that the second-order equation $ay''+by'+cy=0$, a, b, and c constants, always possesses at least one solution of the form $y_1=e^{m_1x}$, m_1 a constant.

(b) Explain why the differential equation in part (a) must then have a second solution either of the form $y_2=e^{m_2x}$ or of the form $y_2=xe^{m_1x}$, m_1 and m_2 constants.

(c) Reexamine Problems 1–8. Can you explain why the statements in parts (a) and (b) above are not contradicted by the answers to Problems 3–5?

22. Verify that $y_1(x)=x$ is a solution of $xy''-xy'+y=0$. Use reduction of order to find a second solution $y_2(x)$ in the form of an infinite series. Conjecture an interval of definition for $y_2(x)$.

Computer Lab Assignments

23. **(a)** Verify that $y_1(x)=e^x$ is a solution of

$$xy''-(x+10)y'+10y=0.$$

(b) Use (5) to find a second solution $y_2(x)$. Use a CAS to carry out the required integration.

(c) Explain, using Corollary (A) of Theorem 4.1.2, why the second solution can be written compactly as

$$y_2(x)=\sum_{n=0}^{10}\frac{1}{n!}x^n.$$

4.3 HOMOGENEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

REVIEW MATERIAL

- Review Problems 27–30 in Exercises 1.1 and Theorem 4.1.5
- Review the algebra of solving polynomial equations (see the *Student Resource Manual*)

INTRODUCTION As a means of motivating the discussion in this section, let us return to first order differential equations—more specificall , to *homogeneous* linear equations $ay'+by=0$, where the coefficients $a\neq 0$ and b are constants. This type of equation can be solved either by separation of variables or with the aid of an integrating factor, but there is another solution method, one that uses only algebra. Before illustrating this alternative method, we make one observation: Solving $ay'+by=0$ for y' yields $y'=ky$, where k is a constant. This observation reveals the nature of the unknown solution y; the only nontrivial elementary function whose derivative is a constant multiple of itself is an exponential function e^{mx}. Now the new solution method: If we substitute $y=e^{mx}$ and $y'=me^{mx}$ into $ay'+by=0$, we get

$$ame^{mx}+be^{mx}=0\quad\text{or}\quad e^{mx}(am+b)=0.$$

Since e^{mx} is never zero for real values of x, the last equation is satisfied only when m is a solution or root of the first-degree polynomial equation $am+b=0$. For this single value of m, $y=e^{mx}$ is a solution of the DE. To illustrate, consider the constant-coefficient equation $2y'+5y=0$. It is not necessary to go through the differentiation and substitution of $y=e^{mx}$ into the DE; we merely have to form the equation $2m+5=0$ and solve it for m. From $m=-\frac{5}{2}$ we conclude that $y=e^{-5x/2}$ is a solution of $2y'+5y=0$, and its general solution on the interval $(-\infty,\ )$ is $y=c_1e^{-5x/2}$.

In this section we will see that the foregoing procedure can produce exponential solutions for homogeneous linear higher-order DEs,

$$a_ny^{(n)}+a_{n-1}y^{(n-1)}+\cdots+a_2y''+a_1y'+a_0y=0, \tag{1}$$

where the coefficients a_i, $i=0,1,\ldots,n$ are real constants and $a_n\neq 0$.