The last equation is separable. Using partial fractions

$$\frac{du}{(u-3)(u+3)} = dx \qquad \text{or} \qquad \frac{1}{6}\left[\frac{1}{u-3} - \frac{1}{u+3}\right] du = dx$$

and then integrating yields

$$\frac{1}{6}\ln\left|\frac{u-3}{u+3}\right| = x + c_1 \qquad \text{or} \qquad \frac{u-3}{u+3} = e^{6x+6c_1} = ce^{6x}. \qquad \leftarrow \text{replace } e^{6c_1} \text{ by } c$$

Solving the last equation for u and then resubstituting gives the solution

$$u = \frac{3(1+ce^{6x})}{1-ce^{6x}} \qquad \text{or} \qquad y = 2x + \frac{3(1+ce^{6x})}{1-ce^{6x}}. \tag{6}$$

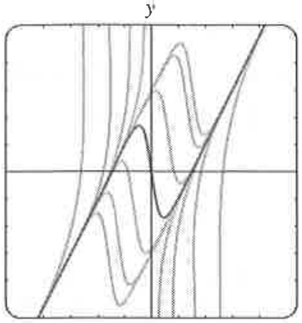

FIGURE 2.5.1 Solutions of DE in Example 3

Finally, applying the initial condition $y(0) = 0$ to the last equation in (6) gives $c = -1$. Figure 2.5.1, obtained with the aid of a graphing utility, shows the graph of the particular solution $y = 2x + \dfrac{3(1-e^{6x})}{1+e^{6x}}$ in dark blue, along with the graphs of some other members of the family of solutions (6). ≡

EXERCISES 2.5

Answers to selected odd-numbered problems begin on page ANS-2.

Each DE in Problems 1–14 is homogeneous.

In Problems 1–10 solve the given differential equation by using an appropriate substitution.

1. $(x - y)\,dx + x\,dy = 0$
2. $(x + y)\,dx + x\,dy = 0$
3. $x\,dx + (y - 2x)\,dy = 0$
4. $y\,dx = 2(x + y)\,dy$
5. $(y^2 + yx)\,dx - x^2\,dy = 0$
6. $(y^2 + yx)\,dx + x^2\,dy = 0$
7. $\dfrac{dy}{dx} = \dfrac{y - x}{y + x}$
8. $\dfrac{dy}{dx} = \dfrac{x + 3y}{3x + y}$
9. $-y\,dx + (x + \sqrt{xy})\,dy = 0$
10. $x\dfrac{dy}{dx} = y + \sqrt{x^2 - y^2}, \quad x > 0$

In Problems 11–14 solve the given initial-value problem.

11. $xy^2\dfrac{dy}{dx} = y^3 - x^3, \quad y(1) = 2$
12. $(x^2 + 2y^2)\dfrac{dx}{dy} = xy, \quad y(-1) = 1$
13. $(x + ye^{y/x})\,dx - xe^{y/x}\,dy = 0, \quad y(1) = 0$
14. $y\,dx + x(\ln x - \ln y - 1)\,dy = 0, \quad y(1) = e$

Each DE in Problems 15–22 is a Bernoulli equation.

In Problems 15–20 solve the given differential equation by using an appropriate substitution.

15. $x\dfrac{dy}{dx} + y = \dfrac{1}{y^2}$
16. $\dfrac{dy}{dx} - y = e^x y^2$
17. $\dfrac{dy}{dx} = y(xy^3 - 1)$
18. $x\dfrac{dy}{dx} - (1 + x)y = xy^2$
19. $t^2\dfrac{dy}{dt} + y^2 = ty$
20. $3(1 + t^2)\dfrac{dy}{dt} = 2ty(y^3 - 1)$

In Problems 21 and 22 solve the given initial-value problem.

21. $x^2\dfrac{dy}{dx} - 2xy = 3y^4, \quad y(1) = \frac{1}{2}$
22. $y^{1/2}\dfrac{dy}{dx} + y^{3/2} = 1, \quad y(0) = 4$

Each DE in Problems 23–30 is of the form given in (5).

In Problems 23–28 solve the given differential equation by using an appropriate substitution.

23. $\dfrac{dy}{dx} = (x + y + 1)^2$
24. $\dfrac{dy}{dx} = \dfrac{1 - x - y}{x + y}$
25. $\dfrac{dy}{dx} = \tan^2(x + y)$
26. $\dfrac{dy}{dx} = \sin(x + y)$
27. $\dfrac{dy}{dx} = 2 + \sqrt{y - 2x + 3}$
28. $\dfrac{dy}{dx} = 1 + e^{y-x+5}$

In Problems 29 and 30 solve the given initial-value problem.

29. $\frac{dy}{dx} = \cos(x + y), \quad y(0) = \quad /4$

30. $\frac{dy}{dx} = \frac{3x + 2y}{3x + 2y + 2}, \quad y(-1) = -1$

Discussion Problems

31. Explain why it is always possible to express any homogeneous differential equation $M(x, y)\,dx + N(x, y)\,dy = 0$ in the form

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right).$$

You might start by proving that

$$M(x, y) = x^\alpha M(1, y/x) \quad \text{and} \quad N(x, y) = x^\alpha N(1, y/x).$$

32. Put the homogeneous differential equation

$$(5x^2 - 2y^2)\,dx - xy\,dy = 0$$

into the form given in Problem 31.

33. **(a)** Determine two singular solutions of the DE in Problem 10.

(b) If the initial condition $y(5) = 0$ is as prescribed in Problem 10, then what is the largest interval I over which the solution is defined? Use a graphing utility to graph the solution curve for the IVP.

34. In Example 3 the solution $y(x)$ becomes unbounded as $x \to \pm\infty$. Nevertheless, $y(x)$ is asymptotic to a curve as $x \to -\infty$ and to a different curve as $x \to \infty$. What are the equations of these curves?

35. The differential equation $dy/dx = P(x) + Q(x)y + R(x)y^2$ is known as **Riccati's equation.**

(a) A Riccati equation can be solved by a succession of two substitutions *provided* that we know a particular solution y_1 of the equation. Show that the substitution $y = y_1 + u$ reduces Riccati's equation to a Bernoulli equation (4) with $n = 2$. The Bernoulli equation can then be reduced to a linear equation by the substitution $w = u^{-1}$.

(b) Find a one-parameter family of solutions for the differential equation

$$\frac{dy}{dx} = -\frac{4}{x^2} - \frac{1}{x}y + y^2$$

where $y_1 = 2/x$ is a known solution of the equation.

36. Determine an appropriate substitution to solve

$$xy' = y\ln(xy).$$

Mathematical Models

37. **Falling Chain** In Problem 45 in Exercises 2.4 we saw that a mathematical model for the velocity v of a chain slipping off the edge of a high horizontal platform is

$$xv\frac{dv}{dx} + v^2 = 32x.$$

In that problem you were asked to solve the DE by converting it into an exact equation using an integrating factor. This time solve the DE using the fact that it is a Bernoulli equation.

38. **Population Growth** In the study of population dynamics one of the most famous models for a growing but bounded population is the **logistic equation**

$$\frac{dP}{dt} = P(a - bP),$$

where a and b are positive constants. Although we will come back to this equation and solve it by an alternative method in Section 3.2, solve the DE this first time using the fact that it is a Bernoulli equation.

2.6 A NUMERICAL METHOD

INTRODUCTION A first-order differential equation $dy/dx = f(x, y)$ is a source of information. We started this chapter by observing that we could garner *qualitative* information from a first-orde DE about its solutions even before we attempted to solve the equation. Then in Sections 2.2–2.5 we examined first-order DEs *analytically*—that is, we developed some procedures for obtaining explicit and implicit solutions. But a differential equation can a possess a solution, yet we may not be able to obtain it analytically. So to round out the picture of the different types of analyses of differential equations, we conclude this chapter with a method by which we can "solve" the differential equation *numerically*—this means that the DE is used as the cornerstone of an algorithm for approximating the unknown solution.

In this section we are going to develop only the simplest of numerical methods—a method that utilizes the idea that a tangent line can be used to approximate the values of a function in a small neighborhood of the point of tangency. A more extensive treatment of numerical methods for ordinary differential equations is given in Chapter 9.