on R enables us to say that not only does a solution exist on some interval I_0 containing x_0, but it is the *only* solution satisfying $y(x_0) = y_0$. However, Theorem 1.2.1 does not give any indication of the sizes of intervals I and I_0; *the interval I of definition need not be as wide as the region R, and the interval I_0 of existence and uniqueness may not be as large as I.* The number $h > 0$ that defines the interval I_0: $(x_0 - h, x_0 + h)$ could be very small, so it is best to think that the solution $y(x)$ is *unique in a local sense* —that is, a solution defined near the point (x_0, y_0). See Problem 50 in Exercises 1.2.

REMARKS

(*i*) The conditions in Theorem 1.2.1 are sufficient but not necessary. This means that when $f(x, y)$ and $\partial f/\partial y$ are continuous on a rectangular region R, it must always follow that a solution of (2) exists and is unique whenever (x_0, y_0) is a point interior to R. However, if the conditions stated in the hypothesis of Theorem 1.2.1 do not hold, then anything could happen: Problem (2) *may* still have a solution and this solution *may* be unique, or (2) may have several solutions, or it may have no solution at all. A rereading of Example 5 reveals that the hypotheses of Theorem 1.2.1 do not hold on the line $y = 0$ for the differential equation $dy/dx = xy^{1/2}$, so it is not surprising, as we saw in Example 4 of this section, that there are two solutions defined on a common interval $-h < x < h$ satisfying $y(0) = 0$. On the other hand, the hypotheses of Theorem 1.2.1 do not hold on the line $y = 1$ for the differential equation $dy/dx = |y - 1|$. Nevertheless it can be proved that the solution of the initial-value problem $dy/dx = |y - 1|$, $y(0) = 1$, is unique. Can you guess this solution?

(*ii*) You are encouraged to read, think about, work, and then keep in mind Problem 49 in Exercises 1.2.

(*iii*) Initial conditions are prescribed at a *single* point x_0. But we are also interested in solving differential equations that are subject to conditions specifie on $y(x)$ or its derivative at *two* different points x_0 and x_1. Conditions such as

$$y(1) = 0, \quad y(5) = 0 \qquad \text{or} \qquad y(\pi/2) = 0, \quad y'(\pi) = 1$$

and called **boundary conditions**. A differential equation together with boundary conditions is called a **boundary-value problem (BVP)**. For example,

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(\pi) = 0$$

is a boundary-value problem. See Problems 39–44 in Exercises 1.2.

When we start to solve differential equations in Chapter 2 we will solve only first-order equations and first-order initial-value problems. The mathematical description of many problems in science and engineering involve second-order IVPs or two-point BVPs. We will examine some of these problems in Chapters 4 and 5.

EXERCISES 1.2

Answers to selected odd-numbered problems begin on page ANS-1.

In Problems 1 and 2, $y = 1/(1 + c_1e^{-x})$ is a one-parameter family of solutions of the first-order DE $y' = y - y^2$. Find a solution of the first-order IVP consisting of this differential equation and the given initial condition.

1. $y(0) = -\frac{1}{3}$ **2.** $y(-1) = 2$

In Problems 3–6, $y = 1/(x^2 + c)$ is a one-parameter family of solutions of the first-order DE $y' + 2xy^2 = 0$. Find a solution of the first-order IVP consisting of this differential equation and the given initial condition. Give the largest interval I over which the solution is defined

3. $y(2) = \frac{1}{3}$ **4.** $y(-2) = \frac{1}{2}$

5. $y(0) = 1$ **6.** $y\left(\frac{1}{2}\right) = -4$

In Problems 7–10, $x = c_1 \cos t + c_2 \sin t$ is a two-parameter family of solutions of the second-order DE $x'' + x = 0$. Find

a solution of the second-order IVP consisting of this differential equation and the given initial conditions.

7. $x(0) = -1, \quad x'(0) = 8$

8. $x(\pi/2) = 0, \quad x'(\pi/2) = 1$

9. $x(\pi/6) = \frac{1}{2}, \quad x'(\pi/6) = 0$

10. $x(\pi/4) = \sqrt{2}, \quad x'(\pi/4) = 2\sqrt{2}$

In Problems 11–14, $y = c_1e^x + c_2e^{-x}$ is a two-parameter family of solutions of the second-order DE $y'' - y = 0$. Find a solution of the second-order IVP consisting of this differential equation and the given initial conditions.

11. $y(0) = 1, \quad y'(0) = 2$

12. $y(1) = 0, \quad y'(1) = e$

13. $y(-1) = 5, \quad y'(-1) = -5$

14. $y(0) = 0, \quad y'(0) = 0$

In Problems 15 and 16 determine by inspection at least two solutions of the given first-order IV .

15. $y' = 3y^{2/3}, \quad y(0) = 0$

16. $xy' = 2y, \quad y(0) = 0$

In Problems 17–24 determine a region of the xy-plane for which the given differential equation would have a unique solution whose graph passes through a point (x_0, y_0) in the region.

17. $\dfrac{dy}{dx} = y^{2/3}$

18. $\dfrac{dy}{dx} = \sqrt{xy}$

19. $x\dfrac{dy}{dx} = y$

20. $\dfrac{dy}{dx} - y = x$

21. $(4 - y^2)y' = x^2$

22. $(1 + y^3)y' = x^2$

23. $(x^2 + y^2)y' = y^2$

24. $(y - x)y' = y + x$

In Problems 25–28 determine whether Theorem 1.2.1 guarantees that the differential equation $y' = \sqrt{y^2 - 9}$ possesses a unique solution through the given point.

25. $(1, 4)$

26. $(5, 3)$

27. $(2, -3)$

28. $(-1, 1)$

29. **(a)** By inspection find a one-parameter family of solutions of the differential equation $xy' = y$. Verify that each member of the family is a solution of the initial-value problem $xy' = y, y(0) = 0$.

(b) Explain part (a) by determining a region R in the xy-plane for which the differential equation $xy' = y$ would have a unique solution through a point (x_0, y_0) in R.

(c) Verify that the piecewise-defined functio

$$y = \begin{cases} 0, & x < 0 \\ x, & x \geq 0 \end{cases}$$

satisfies the condition $y(0) = 0$. Determine whether this function is also a solution of the initial-value problem in part (a).

30. **(a)** Verify that $y = \tan(x + c)$ is a one-parameter family of solutions of the differential equation $y' = 1 + y^2$.

(b) Since $f(x, y) = 1 + y^2$ and $\partial f/\partial y = 2y$ are continuous everywhere, the region R in Theorem 1.2.1 can be taken to be the entire xy-plane. Use the family of solutions in part (a) to find an explicit solution of the first-order initial-value problem $y' = 1 + y^2$, $y(0) = 0$. Even though $x_0 = 0$ is in the interval $(-2, 2)$, explain why the solution is not defined on this interval.

(c) Determine the largest interval I of definition for the solution of the initial-value problem in part (b).

31. **(a)** Verify that $y = -1/(x + c)$ is a one-parameter family of solutions of the differential equation $y' = y^2$.

(b) Since $f(x, y) = y^2$ and $\partial f/\partial y = 2y$ are continuous everywhere, the region R in Theorem 1.2.1 can be taken to be the entire xy-plane. Find a solution from the family in part (a) that satisfies $y(0) = 1$. Then find a solution from the family in part (a) that satisfies $y(0) = -1$. Determine the largest interval I of definition for the solution of each initial-value problem.

(c) Determine the largest interval I of definition for the solution of the first-order initial-value problem $y' = y^2, y(0) = 0$. [*Hint*: The solution is not a member of the family of solutions in part (a).]

32. **(a)** Show that a solution from the family in part (a) of Problem 31 that satisfies $y' = y^2, y(1) = 1$, is $y = 1/(2 - x)$.

(b) Then show that a solution from the family in part (a) of Problem 31 that satisfies $y' = y^2, y(3) = -1$, is $y = 1/(2 - x)$.

(c) Are the solutions in parts (a) and (b) the same?

33. **(a)** Verify that $3x^2 - y^2 = c$ is a one-parameter family of solutions of the differential equation $y\,dy/dx = 3x$.

(b) By hand, sketch the graph of the implicit solution $3x^2 - y^2 = 3$. Find all explicit solutions $y = \phi(x)$ of the DE in part (a) defined by this relation. Give the interval I of definition of each explicit solution

(c) The point $(-2, 3)$ is on the graph of $3x^2 - y^2 = 3$, but which of the explicit solutions in part (b) satisfies $y(-2) = 3$?

34. **(a)** Use the family of solutions in part (a) of Problem 33 to find an implicit solution of the initial-value

problem $y\,dy/dx = 3x$, $y(2) = -4$. Then, by hand, sketch the graph of the explicit solution of this problem and give its interval I of definition

(b) Are there any explicit solutions of $y\,dy/dx = 3x$ that pass through the origin?

In Problems 35–38 the graph of a member of a family of solutions of a second-order differential equation $d^2y/dx^2 = f(x, y, y')$ is given. Match the solution curve with at least one pair of the following initial conditions.

(a) $y(1) = 1,\quad y'(1) = -2$
(b) $y(-1) = 0,\quad y'(-1) = -4$
(c) $y(1) = 1,\quad y'(1) = 2$
(d) $y(0) = -1,\quad y'(0) = 2$
(e) $y(0) = -1,\quad y'(0) = 0$
(f) $y(0) = -4,\quad y'(0) = -2$

35.

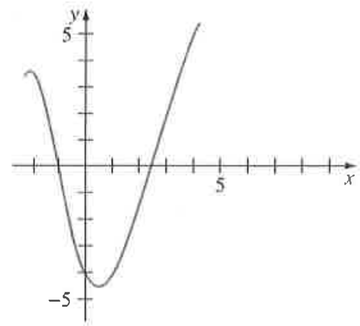

FIGURE 1.2.7 Graph for Problem 35

36.

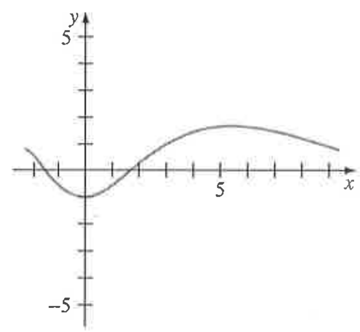

FIGURE 1.2.8 Graph for Problem 36

37.

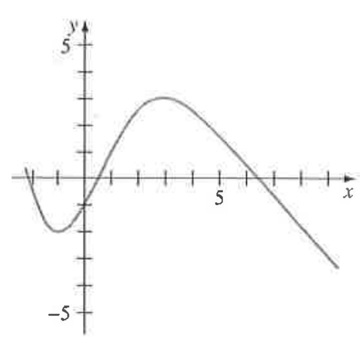

FIGURE 1.2.9 Graph for Problem 37

38.

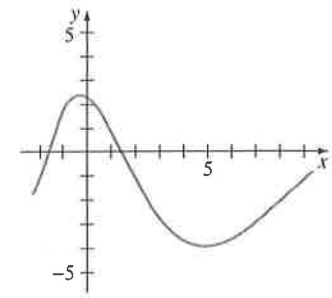

FIGURE 1.2.10 Graph for Problem 38

In Problems 39–44, $y = c_1 \cos 2x + c_2 \sin 2x$ is a two-parameter family of solutions of the second-order DE $y'' + 4y = 0$. If possible, find a solution of the differential equation that satisfies the given side conditions. The conditions specified at two different points are called boundary conditions.

39. $y(0) = 0,\ y(\pi/4) = 3$
40. $y(0) = 0,\ y(\pi) = 0$
41. $y'(0) = 0,\ y'(\pi/6) = 0$
42. $y(0) = 1,\ y'(\pi) = 5$
43. $y(0) = 0,\ y(\pi) = 2$
44. $y'(\pi/2) = 1,\ y'(\pi) = 0$

Discussion Problems

In Problems 45 and 46 use Problem 51 in Exercises 1.1 and (2) and (3) of this section.

45. Find a function $y = f(x)$ whose graph at each point (x, y) has the slope given by $8e^{2x} + 6x$ and has the y-intercept $(0, 9)$.

46. Find a function $y = f(x)$ whose second derivative is $y'' = 12x - 2$ at each point (x, y) on its graph and $y = -x + 5$ is tangent to the graph at the point corresponding to $x = 1$.

47. Consider the initial-value problem $y' = x - 2y$, $y(0) = \frac{1}{2}$. Determine which of the two curves shown in Figure 1.2.11 is the only plausible solution curve. Explain your reasoning.

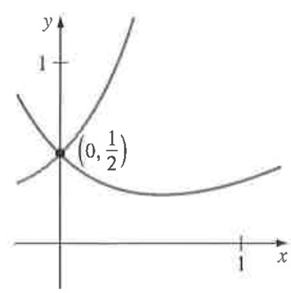

FIGURE 1.2.11 Graphs for Problem 47

48. Determine a plausible value of x_0 for which the graph of the solution of the initial-value problem $y' + 2y = 3x - 6$, $y(x_0) = 0$ is tangent to the x-axis at $(x_0, 0)$. Explain your reasoning.

49. Suppose that the first-order differential equation $dy/dx = f(x, y)$ possesses a one-parameter family of solutions and that $f(x, y)$ satisfies the hypotheses of Theorem 1.2.1 in some rectangular region R of the xy-plane. Explain why two different solution curves cannot intersect or be tangent to each other at a point (x_0, y_0) in R.

50. The functions $y(x) = \frac{1}{16}x^4$, $-\infty < x < \infty$ and

$$y(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{16}x^4, & x \geq 0 \end{cases}$$

have the same domain but are clearly different. See Figures 1.2.12(a) and 1.2.12(b), respectively. Show that both functions are solutions of the initial-value problem $dy/dx = xy^{1/2}$, $y(2) = 1$ on the interval $(-\infty, \infty)$. Resolve the apparent contradiction between this fact and the last sentence in Example 5.

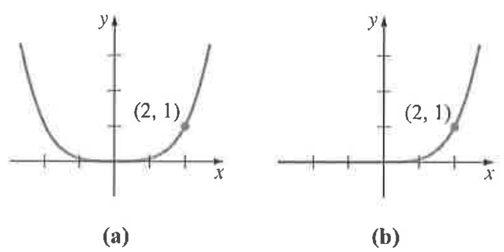

FIGURE 1.2.12 Two solutions of the IVP in Problem 50

Mathematical Model

51. Population Growth Beginning in the next section we will see that differential equations can be used to describe or *model* many different physical systems. In this problem suppose that a model of the growing population of a small community is given by the initial-value problem

$$\frac{dP}{dt} = 0.15P(t) + 20, \quad P(0) = 100,$$

where P is the number of individuals in the community and time t is measured in years. How fast—that is, at what *rate*—is the population increasing at $t = 0$? How fast is the population increasing when the population is 500?

1.3 DIFFERENTIAL EQUATIONS AS MATHEMATICAL MODELS

REVIEW MATERIAL

- Units of measurement for weight, mass, and density
- Newton's second law of motion
- Hooke's law
- Kirchhoff's laws
- Archimedes' principle

INTRODUCTION In this section we introduce the notion of a differential equation as a mathematical model and discuss some specific models in biology, chemistry, and physics. Once we have studied some methods for solving DEs in Chapters 2 and 4, we return to, and solve, some of these models in Chapters 3 and 5.

≡ Mathematical Models It is often desirable to describe the behavior of some real-life system or phenomenon, whether physical, sociological, or even economic, in mathematical terms. The mathematical description of a system of phenomenon is called a **mathematical model** and is constructed with certain goals in mind. For example, we may wish to understand the mechanisms of a certain ecosystem by studying the growth of animal populations in that system, or we may wish to date fossils by analyzing the decay of a radioactive substance, either in the fossil or in the stratum in which it was discovered.