refers to explicit solutions that are expressible in terms of *elementary* (or familiar) *functions:* finite combinations of integer powers of x, roots, exponential and logarithmic functions, and trigonometric and inverse trigonometric functions.

(*vi*) If *every* solution of an nth-order ODE $F(x, y, y', \ldots, y^{(n)}) = 0$ on an interval I can be obtained from an n-parameter family $G(x, y, c_1, c_2, \ldots, c_n) = 0$ by appropriate choices of the parameters c_i, $i = 1, 2, \ldots, n$, we then say that the family is the **general solution** of the DE. In solving linear ODEs, we shall impose relatively simple restrictions on the coefficients of the equation; with these restrictions one can be assured that not only does a solution exist on an interval but also that a family of solutions yields all possible solutions. Nonlinear ODEs, with the exception of some first-order equations, are usually difficult or impossible to solve in terms of elementary functions. Furthermore, if we happen to obtain a family of solutions for a nonlinear equation, it is not obvious whether this family contains all solutions. On a practical level, then, the designation "general solution" is applied only to linear ODEs. Don't be concerned about this concept at this point, but store the words "general solution" in the back of your mind—we will come back to this notion in Section 2.3 and again in Chapter 4.

EXERCISES 1.1

Answers to selected odd-numbered problems begin on page ANS-1.

In Problems 1–8 state the order of the given ordinary differential equation. Determine whether the equation is linear or nonlinear by matching it with (6).

1. $(1 - x)y'' - 4xy' + 5y = \cos x$
2. $x\dfrac{d^3y}{dx^3} - \left(\dfrac{dy}{dx}\right)^4 + y = 0$
3. $t^5y^{(4)} - t^3y'' + 6y = 0$
4. $\dfrac{d^2u}{dr^2} + \dfrac{du}{dr} + u = \cos(r + u)$
5. $\dfrac{d^2y}{dx^2} = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}$
6. $\dfrac{d^2R}{dt^2} = -\dfrac{k}{R^2}$
7. $(\sin\theta)y''' - (\cos\theta)y' = 2$
8. $\ddot{x} - \left(1 - \dfrac{\dot{x}^2}{3}\right)\dot{x} + x = 0$

In Problems 9 and 10 determine whether the given first-order differential equation is linear in the indicated dependent variable by matching it with the first differential equation given in (7).

9. $(y^2 - 1)\,dx + x\,dy = 0$; in y; in x
10. $u\,dv + (v + uv - ue^u)\,du = 0$; in v; in u

In Problems 11–14 verify that the indicated function is an explicit solution of the given differential equation. Assume an appropriate interval I of definition for each solution

11. $2y' + y = 0;\quad y = e^{-x/2}$
12. $\dfrac{dy}{dt} + 20y = 24;\quad y = \dfrac{6}{5} - \dfrac{6}{5}e^{-20t}$
13. $y'' - 6y' + 13y = 0;\quad y = e^{3x}\cos 2x$
14. $y'' + y = \tan x;\quad y = -(\cos x)\ln(\sec x + \tan x)$

In Problems 15–18 verify that the indicated function $y = \phi(x)$ is an explicit solution of the given first-orde differential equation. Proceed as in Example 2, by considering ϕ simply as a *function,* give its domain. Then by considering ϕ as a *solution* of the differential equation, give at least one interval I of definition

15. $(y - x)y' = y - x + 8;\quad y = x + 4\sqrt{x + 2}$
16. $y' = 25 + y^2;\quad y = 5\tan 5x$
17. $y' = 2xy^2;\quad y = 1/(4 - x^2)$
18. $2y' = y^3\cos x;\quad y = (1 - \sin x)^{-1/2}$

In Problems 19 and 20 verify that the indicated expression is an implicit solution of the given first-order differential equation. Find at least one explicit solution $y = \phi(x)$ in each case.

Use a graphing utility to obtain the graph of an explicit solution. Give an interval I of definition of each solution ϕ.

19. $\dfrac{dX}{dt} = (X-1)(1-2X);\quad \ln\left(\dfrac{2X-1}{X-1}\right) = t$

20. $2xy\,dx + (x^2 - y)\,dy = 0;\quad -2x^2y + y^2 = 1$

In Problems 21–24 verify that the indicated family of functions is a solution of the given differential equation. Assume an appropriate interval I of definition for each solution

21. $\dfrac{dP}{dt} = P(1-P);\quad P = \dfrac{c_1e^t}{1+c_1e^t}$

22. $\dfrac{dy}{dx} + 2xy = 1;\quad y = e^{-x^2}\displaystyle\int_0^x e^{t^2}\,dt + c_1e^{-x^2}$

23. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = 0;\quad y = c_1e^{2x} + c_2xe^{2x}$

24. $x^3\dfrac{d^3y}{dx^3} + 2x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = 12x^2;$

$y = c_1x^{-1} + c_2x + c_3x\ln x + 4x^2$

25. Verify that the piecewise-defined functio

$$y = \begin{cases} -x^2, & x < 0 \\ x^2, & x \geq 0 \end{cases}$$

is a solution of the differential equation $xy' - 2y = 0$ on $(-\infty, \infty)$.

26. In Example 5 we saw that $y = \phi_1(x) = \sqrt{25-x^2}$ and $y = \ _2(x) = -\sqrt{25-x^2}$ are solutions of $dy/dx = -x/y$ on the interval $(-5, 5)$. Explain why the piecewise-defined functio

$$y = \begin{cases} \sqrt{25-x^2}, & -5 < x < 0 \\ -\sqrt{25-x^2}, & 0 \leq x < 5 \end{cases}$$

is *not* a solution of the differential equation on the interval $(-5, 5)$.

In Problems 27–30 find values of m so that the function $y = e^{mx}$ is a solution of the given differential equation.

27. $y' + 2y = 0$

28. $5y' = 2y$

29. $y'' - 5y' + 6y = 0$

30. $2y'' + 7y' - 4y = 0$

In Problems 31 and 32 find values of m so that the function $y = x^m$ is a solution of the given differential equation.

31. $xy'' + 2y' = 0$

32. $x^2y'' - 7xy' + 15y = 0$

In Problems 33–36 use the concept that $y = c$, $-\infty < x < \infty$, is a constant function if and only if $y' = 0$ to determine whether the given differential equation possesses constant solutions.

33. $3xy' + 5y = 10$

34. $y' = y^2 + 2y - 3$

35. $(y-1)y' = 1$

36. $y'' + 4y' + 6y = 10$

In Problems 37 and 38 verify that the indicated pair of functions is a solution of the given system of differential equations on the interval $(-\infty, \infty)$.

37. $\dfrac{dx}{dt} = x + 3y$

$\dfrac{dy}{dt} = 5x + 3y;$

$x = e^{-2t} + 3e^{6t},$

$y = -e^{-2t} + 5e^{6t}$

38. $\dfrac{d^2x}{dt^2} = 4y + e^t$

$\dfrac{d^2y}{dt^2} = 4x - e^t;$

$x = \cos 2t + \sin 2t + \frac{1}{5}e^t,$

$y = -\cos 2t - \sin 2t - \frac{1}{5}e^t$

Discussion Problems

39. Make up a differential equation that does not possess any real solutions.

40. Make up a differential equation that you feel confiden possesses only the trivial solution $y = 0$. Explain your reasoning.

41. What function do you know from calculus is such that its first derivative is itself? Its first derivative is a constant multiple k of itself? Write each answer in the form of a first-order differential equation with a solution.

42. What function (or functions) do you know from calculus is such that its second derivative is itself? Its second derivative is the negative of itself? Write each answer in the form of a second-order differential equation with a solution.

43. Given that $y = \sin x$ is an explicit solution of the first-order differential equation $\dfrac{dy}{dx} = \sqrt{1-y^2}$. Find an interval I of definition [*Hint: I* is *not* the interval $(-\infty, \infty)$.]

44. Discuss why it makes intuitive sense to presume that the linear differential equation $y'' + 2y' + 4y = 5\sin t$ has a solution of the form $y = A\sin t + B\cos t$, where A and B are constants. Then find specific constants A and B so that $y = A\sin t + B\cos t$ is a particular solution of the DE.

In Problems 45 and 46 the given figure represents the graph of an implicit solution $G(x, y) = 0$ of a differential equation $dy/dx = f(x, y)$. In each case the relation $G(x, y) = 0$ implicitly defines several solutions of the DE. Carefully reproduce each figure on a piece of paper. Use different colored pencils to mark off segments, or pieces, on each graph that correspond to graphs of solutions. Keep in mind that a solution ϕ must be a function and differentiable. Use the solution curve to estimate an interval I of definition of each solution ϕ.

45.

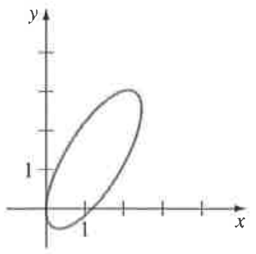

FIGURE 1.1.6 Graph for Problem 45

46.

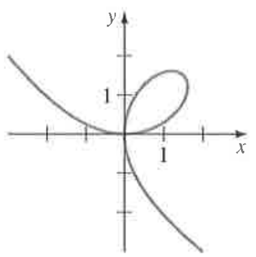

FIGURE 1.1.7 Graph for Problem 46

47. The graphs of members of the one-parameter family $x^3 + y^3 = 3cxy$ are called **folia of Descartes.** Verify that this family is an implicit solution of the first-orde differential equation

$$\frac{dy}{dx} = \frac{y(y^3 - 2x^3)}{x(2y^3 - x^3)}.$$

48. The graph in Figure 1.1.7 is the member of the family of folia in Problem 47 corresponding to $c = 1$. Discuss: How can the DE in Problem 47 help in finding points on the graph of $x^3 + y^3 = 3xy$ where the tangent line is vertical? How does knowing where a tangent line is vertical help in determining an interval I of definitio of a solution ϕ of the DE? Carry out your ideas, and compare with your estimates of the intervals in Problem 46.

49. In Example 5 the largest interval I over which the explicit solutions $y = \phi_1(x)$ and $y = \phi_2(x)$ are define is the open interval $(-5, 5)$. Why can't the interval I of definition be the closed interval $-5, 5]$?

50. In Problem 21 a one-parameter family of solutions of the DE $P' = P(1 - P)$ is given. Does any solution curve pass through the point $(0, 3)$? Through the point $(0, 1)$?

51. Discuss, and illustrate with examples, how to solve differential equations of the forms $dy/dx = f(x)$ and $d^2y/dx^2 = f(x)$.

52. The differential equation $x(y')^2 - 4y' - 12x^3 = 0$ has the form given in (4). Determine whether the equation can be put into the normal form $dy/dx = f(x, y)$.

53. The normal form (5) of an nth-order differential equation is equivalent to (4) whenever both forms have exactly the same solutions. Make up a first-order differential equation for which $F(x, y, y') = 0$ is not equivalent to the normal form $dy/dx = f(x, y)$.

54. Find a linear second-order differential equation $F(x, y, y', y'') = 0$ for which $y = c_1x + c_2x^2$ is a two-parameter family of solutions. Make sure that your equation is free of the arbitrary parameters c_1 and c_2.

Qualitative information about a solution $y = \phi(x)$ of a differential equation can often be obtained from the equation itself. Before working Problems 55–58, recall the geometric significance of the derivatives dy/dx and d^2y/dx^2.

55. Consider the differential equation $dy/dx = e^{-x^2}$.

(a) Explain why a solution of the DE must be an increasing function on any interval of the x-axis.

(b) What are $\lim_{x \to -\infty} dy/dx$ and $\lim_{x \to \infty} dy/dx$? What does this suggest about a solution curve as $x \to \pm\infty$?

(c) Determine an interval over which a solution curve is concave down and an interval over which the curve is concave up.

(d) Sketch the graph of a solution $y = \phi(x)$ of the differential equation whose shape is suggested by parts (a)–(c).

56. Consider the differential equation $dy/dx = 5 - y$.

(a) Either by inspection or by the method suggested in Problems 33–36, find a constant solution of the DE.

(b) Using only the differential equation, find intervals on the y-axis on which a nonconstant solution $y = \phi(x)$ is increasing. Find intervals on the y-axis on which $y = \phi(x)$ is decreasing.

57. Consider the differential equation $dy/dx = y(a - by)$, where a and b are positive constants.

(a) Either by inspection or by the method suggested in Problems 33–36, find two constant solutions of the DE.

(b) Using only the differential equation, find intervals on the y-axis on which a nonconstant solution $y = \phi(x)$ is increasing. Find intervals on which $y = \phi(x)$ is decreasing.

(c) Using only the differential equation, explain why $y = a/2b$ is the y-coordinate of a point of inflectio of the graph of a nonconstant solution $y = \phi(x)$.

(d) On the same coordinate axes, sketch the graphs of the two constant solutions found in part (a). These constant solutions partition the xy-plane into three regions. In each region, sketch the graph of a non-constant solution $y = \phi(x)$ whose shape is suggested by the results in parts (b) and (c).

58. Consider the differential equation $y' = y^2 + 4$.

(a) Explain why there exist no constant solutions of the DE.

(b) Describe the graph of a solution $y = \phi(x)$. For example, can a solution curve have any relative extrema?

(c) Explain why $y = 0$ is the y-coordinate of a point of inflection of a solution curve

(d) Sketch the graph of a solution $y = \phi(x)$ of the differential equation whose shape is suggested by parts (a)–(c).

Computer Lab Assignments

In Problems 59 and 60 use a CAS to compute all derivatives and to carry out the simplifications needed to verify that the indicated function is a particular solution of the given differential equation.

59. $y^{(4)} - 20y''' + 158y'' - 580y' + 841y = 0;$
$y = xe^{5x}\cos 2x$

60. $x^3y''' + 2x^2y'' + 20xy' - 78y = 0;$
$y = 20\dfrac{\cos(5\ln x)}{x} - 3\dfrac{\sin(5\ln x)}{x}$

1.2 INITIAL-VALUE PROBLEMS

REVIEW MATERIAL

- Normal form of a DE
- Solution of a DE
- Family of solutions

INTRODUCTION We are often interested in problems in which we seek a solution $y(x)$ of a differential equation so that $y(x)$ also satisfies certain prescribed side conditions—that is, conditions that are imposed on the unknown function $y(x)$ and its derivatives at a point x_0. On some interval I containing x_0 the problem of solving an nth-order differential equation subject to n side conditions specified at x_0:

$$\begin{aligned} &\textit{Solve:} && \frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)}) \\ &\textit{Subject to:} && y(x_0) = y_0,\ y'(x_0) = y_1,\ \ldots,\ y^{(n-1)}(x_0) = y_{n-1}, \end{aligned} \tag{1}$$

where $y_0, y_1, \ldots, y_{n-1}$ are arbitrary real constants, is called an ***n*th-order initial-value problem (IVP).** The values of $y(x)$ and its first $n - 1$ derivatives at x_0, $y(x_0) = y_0, y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1}$ are called **initial conditions (IC).**

Solving an nth-order initial-value problem such as (1) frequently entails first finding an n-parameter family of solutions of the given differential equation and then using the initial-conditions at x_0 to determine the n constants in this family. The resulting particular solution is defined on some interval I containing the initial point x_0.

≡ Geometric Interpretation of IVPs The cases $n = 1$ and $n = 2$ in (1),

$$\begin{aligned} &\textit{Solve:} && \frac{dy}{dx} = f(x, y) \\ &\textit{Subject to:} && y(x_0) = y_0 \end{aligned} \tag{2}$$