53 (14.5)

$z = f(x,y)$
$x = r\cos\theta$
$y = r\sin\theta$

<u>Show:</u> $$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} + \frac{1}{r}\frac{\partial z}{\partial r}$$

<u>Find Terms in r.h.s:</u>

(1) $\frac{\partial^2 z}{\partial r^2} = ?$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta$$

$$\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial r}\right) = \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\cdot\cos\theta + \frac{\partial z}{\partial y}\cdot\sin\theta\right)$$

$$\frac{\partial}{\partial r} = \frac{\partial}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\frac{\partial y}{\partial r}$$

so $$= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta\right)\cdot\underbrace{\frac{\partial x}{\partial r}}_{=\cos\theta} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta\right)\cdot\underbrace{\frac{\partial y}{\partial r}}_{=\sin}$$

$$= \frac{\partial^2 z}{\partial x^2}\cos^2\theta + \underbrace{\frac{\partial^2 z}{\partial x\partial y}\sin\theta\cos\theta + \frac{\partial^2 z}{\partial y\partial x}\cos\theta\sin\theta}_{} + \frac{\partial^2 z}{\partial y^2}\sin^2$$

$$= \frac{\partial^2 z}{\partial x^2}\cos^2\theta + 2\sin\theta\cdot\cos\theta\cdot\frac{\partial^2 z}{\partial x\partial y} + \frac{\partial^2 z}{\partial y^2}\sin^2\theta$$

(2) $\frac{\partial^2 z}{\partial \theta^2} = ?$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} = \underbrace{-\frac{\partial z}{\partial x}r\sin\theta + \frac{\partial z}{\partial y}r\cos\theta}_{}$$

$$\frac{\partial^2 z}{\partial \theta^2} = \frac{\partial}{\partial \theta}(\nearrow)$$

$$= \frac{\partial}{\partial x}(\quad)\underbrace{\frac{\partial x}{\partial \theta}}_{-r\sin\theta} + \frac{\partial}{\partial y}(\quad)\underbrace{\frac{\partial y}{\partial \theta}}_{r\cos\theta}$$

$$= -\frac{\partial^2 z}{\partial x^2}r\sin\theta(-r\sin\theta) + \frac{\partial^2 z}{\partial x\partial y}r\cos\theta(-r\sin\theta)$$

Use product rule! on $\frac{\partial}{\partial\theta}\left(-\frac{\partial z}{\partial x}\cdot r\sin\theta\right)$ → $$- r\frac{\partial z}{\partial x}\cos\theta$$

$$+ \frac{\partial^2 z}{\partial y^2}r\cos\theta(r\cos\theta) + \frac{\partial^2 z}{\partial x\partial y}(-r\sin\theta)(r\cos\theta)$$

and $\frac{\partial}{\partial\theta}\left(\frac{\partial z}{\partial r}r\cos\theta\right)$ → $$- r\frac{\partial z}{\partial y}\sin\theta$$

Put r.h.s all together:

$$\frac{\partial^2 z}{\partial x^2}\cos^2\theta + 2\sin\theta.\cos\theta.\frac{\partial^2 z}{\partial x\partial y} + \frac{\partial^2 z}{\partial y^2}\sin^2\theta \qquad \left(\leftarrow \frac{\partial^2 z}{\partial r^2}\right)$$

$$+ \frac{\partial^2 z}{\partial x^2}\sin^2\theta - 2\sin\theta\cos\theta\frac{\partial^2 z}{\partial x\partial y} + \frac{\partial^2 z}{\partial y^2}\cos^2\theta \qquad \left(\leftarrow \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}\right)$$

$$- \frac{1}{r}\cos\theta\frac{\partial z}{\partial x} - \frac{1}{r}\frac{\partial z}{\partial y}\sin\theta$$

$$+ \frac{1}{r}\cos\theta\frac{\partial z}{\partial x} + \frac{1}{r}\frac{\partial z}{\partial y}\sin\theta \qquad \left(\leftarrow \frac{1}{r}\frac{\partial z}{\partial r}\right)$$

$$= \frac{\partial^2 z}{\partial x^2}\left(\cos^2\theta + \sin^2\theta\right) + \frac{\partial^2 z}{\partial y^2}\left(\sin^2\theta + \cos^2\theta\right)$$

$$= \boxed{\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}} \checkmark$$

ouf!