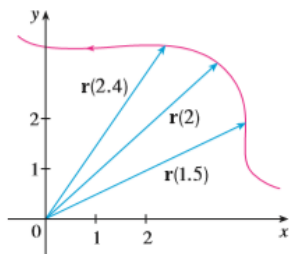


13.4 Exercises

1. The table gives coordinates of a particle moving through space along a smooth curve.
- Find the average velocities over the time intervals $[0, 1]$, $[0.5, 1]$, $[1, 2]$, and $[1, 1.5]$.
 - Estimate the velocity and speed of the particle at $t = 1$.

t	x	y	z
0	2.7	9.8	3.7
0.5	3.5	7.2	3.3
1.0	4.5	6.0	3.0
1.5	5.9	6.4	2.8
2.0	7.3	7.8	2.7

2. The figure shows the path of a particle that moves with position vector $\mathbf{r}(t)$ at time t .
- Draw a vector that represents the average velocity of the particle over the time interval $2 \leq t \leq 2.4$.
 - Draw a vector that represents the average velocity over the time interval $1.5 \leq t \leq 2$.
 - Write an expression for the velocity vector $\mathbf{v}(2)$.
 - Draw an approximation to the vector $\mathbf{v}(2)$ and estimate the speed of the particle at $t = 2$.



- 3–8 Find the velocity, acceleration, and speed of a particle with the given position function. Sketch the path of the particle and draw the velocity and acceleration vectors for the specified value of t .

3. $\mathbf{r}(t) = \langle -\frac{1}{2}t^2, t \rangle$, $t = 2$

4. $\mathbf{r}(t) = \langle 2 - t, 4\sqrt{t} \rangle$, $t = 1$

5. $\mathbf{r}(t) = 3 \cos t \mathbf{i} + 2 \sin t \mathbf{j}$, $t = \pi/3$

6. $\mathbf{r}(t) = e^t \mathbf{i} + e^{2t} \mathbf{j}$, $t = 0$

7. $\mathbf{r}(t) = t \mathbf{i} + t^2 \mathbf{j} + 2 \mathbf{k}$, $t = 1$

8. $\mathbf{r}(t) = t \mathbf{i} + 2 \cos t \mathbf{j} + \sin t \mathbf{k}$, $t = 0$

- 9–14 Find the velocity, acceleration, and speed of a particle with the given position function.

9. $\mathbf{r}(t) = \langle t^2 + t, t^2 - t, t^3 \rangle$ 10. $\mathbf{r}(t) = \langle 2 \cos t, 3t, 2 \sin t \rangle$

11. $\mathbf{r}(t) = \sqrt{2} t \mathbf{i} + e^t \mathbf{j} + e^{-t} \mathbf{k}$ 12. $\mathbf{r}(t) = t^2 \mathbf{i} + 2t \mathbf{j} + \ln t \mathbf{k}$

13. $\mathbf{r}(t) = e^t(\cos t \mathbf{i} + \sin t \mathbf{j} + t \mathbf{k})$

14. $\mathbf{r}(t) = \langle t^2, \sin t - t \cos t, \cos t + t \sin t \rangle$, $t \geq 0$

- 15–16 Find the velocity and position vectors of a particle that has the given acceleration and the given initial velocity and position.

15. $\mathbf{a}(t) = \mathbf{i} + 2 \mathbf{j}$, $\mathbf{v}(0) = \mathbf{k}$, $\mathbf{r}(0) = \mathbf{i}$

16. $\mathbf{a}(t) = 2 \mathbf{i} + 6t \mathbf{j} + 12t^2 \mathbf{k}$, $\mathbf{v}(0) = \mathbf{i}$, $\mathbf{r}(0) = \mathbf{j} - \mathbf{k}$

17–18

- (a) Find the position vector of a particle that has the given acceleration and the specified initial velocity and position.

- (b) Use a computer to graph the path of the particle.

17. $\mathbf{a}(t) = 2t \mathbf{i} + \sin t \mathbf{j} + \cos 2t \mathbf{k}$, $\mathbf{v}(0) = \mathbf{i}$, $\mathbf{r}(0) = \mathbf{j}$

18. $\mathbf{a}(t) = t \mathbf{i} + e^t \mathbf{j} + e^{-t} \mathbf{k}$, $\mathbf{v}(0) = \mathbf{k}$, $\mathbf{r}(0) = \mathbf{j} + \mathbf{k}$

19. The position function of a particle is given by $\mathbf{r}(t) = \langle t^2, 5t, t^2 - 16t \rangle$. When is the speed a minimum?

20. What force is required so that a particle of mass m has the position function $\mathbf{r}(t) = t^3 \mathbf{i} + t^2 \mathbf{j} + t^3 \mathbf{k}$?

21. A force with magnitude 20 N acts directly upward from the xy -plane on an object with mass 4 kg. The object starts at the origin with initial velocity $\mathbf{v}(0) = \mathbf{i} - \mathbf{j}$. Find its position function and its speed at time t .

22. Show that if a particle moves with constant speed, then the velocity and acceleration vectors are orthogonal.

23. A projectile is fired with an initial speed of 200 m/s and angle of elevation 60° . Find (a) the range of the projectile, (b) the maximum height reached, and (c) the speed at impact.

24. Rework Exercise 23 if the projectile is fired from a position 100 m above the ground.

25. A ball is thrown at an angle of 45° to the ground. If the ball lands 90 m away, what was the initial speed of the ball?

26. A gun is fired with angle of elevation 30° . What is the muzzle speed if the maximum height of the shell is 500 m?

27. A gun has muzzle speed 150 m/s. Find two angles of elevation that can be used to hit a target 800 m away.

28. A batter hits a baseball 3 ft above the ground toward the center field fence, which is 10 ft high and 400 ft from home plate. The ball leaves the bat with speed 115 ft/s at an angle 50° above the horizontal. Is it a home run? (In other words, does the ball clear the fence?)
29. A medieval city has the shape of a square and is protected by walls with length 500 m and height 15 m. You are the commander of an attacking army and the closest you can get to the wall is 100 m. Your plan is to set fire to the city by catapulting heated rocks over the wall (with an initial speed of 80 m/s). At what range of angles should you tell your men to set the catapult? (Assume the path of the rocks is perpendicular to the wall.)
30. Show that a projectile reaches three-quarters of its maximum height in half the time needed to reach its maximum height.
31. A ball is thrown eastward into the air from the origin (in the direction of the positive x -axis). The initial velocity is $50\mathbf{i} + 80\mathbf{k}$, with speed measured in feet per second. The spin of the ball results in a southward acceleration of 4 ft/s^2 , so the acceleration vector is $\mathbf{a} = -4\mathbf{j} - 32\mathbf{k}$. Where does the ball land and with what speed?
32. A ball with mass 0.8 kg is thrown southward into the air with a speed of 30 m/s at an angle of 30° to the ground. A west wind applies a steady force of 4 N to the ball in an easterly direction. Where does the ball land and with what speed?
33. Water traveling along a straight portion of a river normally flows fastest in the middle, and the speed slows to almost zero at the banks. Consider a long straight stretch of river flowing north, with parallel banks 40 m apart. If the maximum water speed is 3 m/s, we can use a quadratic function as a basic model for the rate of water flow x units from the west bank: $f(x) = \frac{3}{200}x(40 - x)$.

- (b) If a particle moves with constant speed along a curve, what can you say about its acceleration vector?

37–42 Find the tangential and normal components of the acceleration vector.

37. $\mathbf{r}(t) = (3t - t^3)\mathbf{i} + 3t^2\mathbf{j}$

38. $\mathbf{r}(t) = (1 + t)\mathbf{i} + (t^2 - 2t)\mathbf{j}$

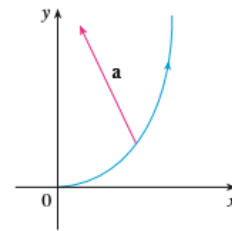
39. $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$

40. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + 3t\mathbf{k}$

41. $\mathbf{r}(t) = e^t\mathbf{i} + \sqrt{2}t\mathbf{j} + e^{-t}\mathbf{k}$

42. $\mathbf{r}(t) = t\mathbf{i} + \cos^2 t\mathbf{j} + \sin^2 t\mathbf{k}$

43. The magnitude of the acceleration vector \mathbf{a} is 10 cm/s^2 . Use the figure to estimate the tangential and normal components of \mathbf{a} .



44. If a particle with mass m moves with position vector $\mathbf{r}(t)$, then its **angular momentum** is defined as $\mathbf{L}(t) = m\mathbf{r}(t) \times \mathbf{v}(t)$ and its **torque** as $\boldsymbol{\tau}(t) = m\mathbf{r}(t) \times \mathbf{a}(t)$. Show that $\mathbf{L}'(t) = \boldsymbol{\tau}(t)$. Deduce that if $\boldsymbol{\tau}(t) = \mathbf{0}$ for all t , then $\mathbf{L}(t)$ is constant. (This is the *law of conservation of angular momentum*.)

33. Water traveling along a straight portion of a river normally flows fastest in the middle, and the speed slows to almost zero at the banks. Consider a long straight stretch of river flowing north, with parallel banks 40 m apart. If the maximum water speed is 3 m/s, we can use a quadratic function as a basic model for the rate of water flow x units from the west bank: $f(x) = \frac{3}{400}x(40 - x)$.
- (a) A boat proceeds at a constant speed of 5 m/s from a point A on the west bank while maintaining a heading perpendicular to the bank. How far down the river on the opposite bank will the boat touch shore? Graph the path of the boat.
- (b) Suppose we would like to pilot the boat to land at the point B on the east bank directly opposite A . If we maintain a constant speed of 5 m/s and a constant heading, find the angle at which the boat should head. Then graph the actual path the boat follows. Does the path seem realistic?
34. Another reasonable model for the water speed of the river in Exercise 33 is a sine function: $f(x) = 3 \sin(\pi x/40)$. If a boater would like to cross the river from A to B with constant heading and a constant speed of 5 m/s, determine the angle at which the boat should head.
35. A particle has position function $\mathbf{r}(t)$. If $\mathbf{r}'(t) = \mathbf{c} \times \mathbf{r}(t)$, where \mathbf{c} is a constant vector, describe the path of the particle.
36. (a) If a particle moves along a straight line, what can you say about its acceleration vector?

44. If a particle with mass m moves with position vector $\mathbf{r}(t)$, then its **angular momentum** is defined as $\mathbf{L}(t) = m\mathbf{r}(t) \times \mathbf{v}(t)$ and its **torque** as $\boldsymbol{\tau}(t) = m\mathbf{r}(t) \times \mathbf{a}(t)$. Show that $\mathbf{L}'(t) = \boldsymbol{\tau}(t)$. Deduce that if $\boldsymbol{\tau}(t) = \mathbf{0}$ for all t , then $\mathbf{L}(t)$ is constant. (This is the *law of conservation of angular momentum*.)

45. The position function of a spaceship is

$$\mathbf{r}(t) = (3 + t)\mathbf{i} + (2 + \ln t)\mathbf{j} + \left(7 - \frac{4}{t^2 + 1}\right)\mathbf{k}$$

and the coordinates of a space station are $(6, 4, 9)$. The captain wants the spaceship to coast into the space station. When should the engines be turned off?

46. A rocket burning its onboard fuel while moving through space has velocity $\mathbf{v}(t)$ and mass $m(t)$ at time t . If the exhaust gases escape with velocity \mathbf{v}_e relative to the rocket, it can be deduced from Newton's Second Law of Motion that

$$m \frac{d\mathbf{v}}{dt} = \frac{dm}{dt} \mathbf{v}_e$$

- (a) Show that $\mathbf{v}(t) = \mathbf{v}(0) - \ln \frac{m(0)}{m(t)} \mathbf{v}_e$.
- (b) For the rocket to accelerate in a straight line from rest to twice the speed of its own exhaust gases, what fraction of its initial mass would the rocket have to burn as fuel?