11.8 Exercises

1. What is a power series?

2. (a) What is the radius of convergence of a power series? How do you find it?
   (b) What is the interval of convergence of a power series? How do you find it?

3–28 Find the radius of convergence and interval of convergence of the series.

3. $\sum_{n=1}^{\infty} (-1)^n n x^n$

4. $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\sqrt[3]{n}}$

5. $\sum_{n=1}^{\infty} \frac{x^n}{2n-1}$

6. $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n^2}$

7. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

8. $\sum_{n=1}^{\infty} n^n x^n$

9. $\sum_{n=1}^{\infty} (-1)^n \frac{n^2 x^n}{2^n}$

10. $\sum_{n=1}^{\infty} \frac{10^n x^n}{n^3}$

11. $\sum_{n=1}^{\infty} \frac{(-3)^n}{n\sqrt{n}} x^n$

12. $\sum_{n=1}^{\infty} \frac{x^n}{n3^n}$

13. $\sum_{n=2}^{\infty} (-1)^n \frac{x^n}{4^n \ln n}$

14. $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

15. $\sum_{n=0}^{\infty} \frac{(x-2)^n}{n^2+1}$

16. $\sum_{n=0}^{\infty} (-1)^n \frac{(x-3)^n}{2n+1}$

17. $\sum_{n=1}^{\infty} \frac{3^n (x+4)^n}{\sqrt{n}}$

18. $\sum_{n=1}^{\infty} \frac{n}{4^n} (x+1)^n$

19. $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^n}$

20. $\sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^n \sqrt{n}}$

Graphing calculator or computer required | CAS Computer algebra system required | 1. Homework Hints available at stewartcalculus.com

21. $\sum_{n=1}^{\infty} \frac{n}{b^n} (x-a)^n, \quad b > 0$

22. $\sum_{n=2}^{\infty} \frac{b^n}{\ln n} (x-a)^n, \quad b > 0$

23. $\sum_{n=1}^{\infty} n!(2x-1)^n$

24. $\sum_{n=1}^{\infty} \frac{n^2 x^n}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)}$

25. $\sum_{n=1}^{\infty} \frac{(5x-4)^n}{n^3}$

26. $\sum_{n=2}^{\infty} \frac{x^{2n}}{n(\ln n)^2}$

27. $\sum_{n=1}^{\infty} \frac{x^n}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}$

28. $\sum_{n=1}^{\infty} \frac{n! x^n}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}$

35. The function J_1 defined by

$$J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(n+1)! 2^{2n+1}}$$

is called the *Bessel function of order 1*.
(a) Find its domain.
(b) Graph the first several partial sums on a common screen.
(c) If your CAS has built-in Bessel functions, graph J_1 on the same screen as the partial sums in part (b) and observe how the partial sums approximate J_1.

36. The function A defined by

$$A(x) = 1 + \frac{x^3}{2 \cdot 3} + \frac{x^6}{2 \cdot 3 \cdot 5 \cdot 6} + \frac{x^9}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9} + \cdots$$

11.9 Exercises

1. If the radius of convergence of the power series $\Sigma_{n=0}^{\infty} c_n x^n$ is 10, what is the radius of convergence of the series $\Sigma_{n=1}^{\infty} nc_n x^{n-1}$? Why?

2. Suppose you know that the series $\Sigma_{n=0}^{\infty} b_n x^n$ converges for $|x| < 2$. What can you say about the following series? Why?

$$\sum_{n=0}^{\infty} \frac{b_n}{n+1} x^{n+1}$$

3–10 Find a power series representation for the function and determine the interval of convergence.

3. $f(x) = \dfrac{1}{1+x}$

4. $f(x) = \dfrac{5}{1-4x^2}$

5. $f(x) = \dfrac{2}{3-x}$

6. $f(x) = \dfrac{1}{x+10}$

7. $f(x) = \dfrac{x}{9+x^2}$

8. $f(x) = \dfrac{x}{2x^2+1}$

9. $f(x) = \dfrac{1+x}{1-x}$

10. $f(x) = \dfrac{x^2}{a^3-x^3}$

11–12 Express the function as the sum of a power series by first using partial fractions. Find the interval of convergence.

11. $f(x) = \dfrac{3}{x^2-x-2}$

12. $f(x) = \dfrac{x+2}{2x^2-x-1}$

13. (a) Use differentiation to find a power series representation for

$$f(x) = \frac{1}{(1+x)^2}$$

What is the radius of convergence?

(b) Use part (a) to find a power series for

$$f(x) = \frac{1}{(1+x)^3}$$

(c) Use part (b) to find a power series for

$$f(x) = \frac{x^2}{(1+x)^3}$$

14. (a) Use Equation 1 to find a power series representation for $f(x) = \ln(1-x)$. What is the radius of convergence?
(b) Use part (a) to find a power series for $f(x) = x\ln(1-x)$.
(c) By putting $x = \frac{1}{2}$ in your result from part (a), express $\ln 2$ as the sum of an infinite series.

15–20 Find a power series representation for the function and determine the radius of convergence.

15. $f(x) = \ln(5-x)$

16. $f(x) = x^2 \tan^{-1}(x^3)$

17. $f(x) = \dfrac{x}{(1+4x)^2}$

18. $f(x) = \left(\dfrac{x}{2-x}\right)^3$

19. $f(x) = \dfrac{1+x}{(1-x)^2}$

20. $f(x) = \dfrac{x^2+x}{(1-x)^3}$

33. Use the result of Example 7 to compute arctan 0.2 correct to five decimal places.

34. Show that the function

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

is a solution of the differential equation

$$f''(x) + f(x) = 0$$

35. (a) Show that J_0 (the Bessel function of order 0 given in Example 4) satisfies the differential equation

$$x^2 J_0''(x) + xJ_0'(x) + x^2 J_0(x) = 0$$

(b) Evaluate $\int_0^1 J_0(x)\,dx$ correct to three decimal places.

36. The Bessel function of order 1 is defined by

$$J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!\,(n+1)!\,2^{2n+1}}$$

(a) Show that J_1 satisfies the differential equation

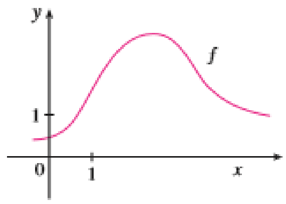
$$x^2 J_1''(x) + xJ_1'(x) + (x^2-1)J_1(x) = 0$$

(b) Show that $J_0'(x) = -J_1(x)$.

11.10 Exercises

1. If $f(x) = \sum_{n=0}^{\infty} b_n(x - 5)^n$ for all x, write a formula for b_8.

2. The graph of f is shown.



(a) Explain why the series

$$1.6 - 0.8(x - 1) + 0.4(x - 1)^2 - 0.1(x - 1)^3 + \cdots$$

is *not* the Taylor series of f centered at 1.

(b) Explain why the series

$$2.8 + 0.5(x - 2) + 1.5(x - 2)^2 - 0.1(x - 2)^3 + \cdots$$

is *not* the Taylor series of f centered at 2.

3. If $f^{(n)}(0) = (n + 1)!$ for $n = 0, 1, 2, \ldots$, find the Maclaurin series for f and its radius of convergence.

4. Find the Taylor series for f centered at 4 if

$$f^{(n)}(4) = \frac{(-1)^n n!}{3^n(n + 1)}$$

What is the radius of convergence of the Taylor series?

5–12 Find the Maclaurin series for $f(x)$ using the definition of a Maclaurin series. [Assume that f has a power series expansion. Do not show that $R_n(x) \to 0$.] Also find the associated radius of convergence.

5. $f(x) = (1 - x)^{-2}$
6. $f(x) = \ln(1 + x)$
7. $f(x) = \sin \pi x$
8. $f(x) = e^{-2x}$
9. $f(x) = 2^x$
10. $f(x) = x \cos x$
11. $f(x) = \sinh x$
12. $f(x) = \cosh x$

13–20 Find the Taylor series for $f(x)$ centered at the given value of a. [Assume that f has a power series expansion. Do not show that $R_n(x) \to 0$.] Also find the associated radius of convergence.

13. $f(x) = x^4 - 3x^2 + 1,\quad a = 1$
14. $f(x) = x - x^3,\quad a = -2$
15. $f(x) = \ln x,\quad a = 2$
16. $f(x) = 1/x,\quad a = -3$

21. Prove that the series obtained in Exercise 7 represents $\sin \pi x$ for all x.

22. Prove that the series obtained in Exercise 18 represents $\sin x$ for all x.

23. Prove that the series obtained in Exercise 11 represents $\sinh x$ for all x.

24. Prove that the series obtained in Exercise 12 represents $\cosh x$ for all x.

25–28 Use the binomial series to expand the function as a power series. State the radius of convergence.

25. $\sqrt[4]{1 - x}$
26. $\sqrt[3]{8 + x}$
27. $\dfrac{1}{(2 + x)^3}$
28. $(1 - x)^{2/3}$

29–38 Use a Maclaurin series in Table 1 to obtain the Maclaurin series for the given function.

29. $f(x) = \sin \pi x$
30. $f(x) = \cos(\pi x/2)$
31. $f(x) = e^x + e^{2x}$
32. $f(x) = e^x + 2e^{-x}$
33. $f(x) = x \cos\left(\frac{1}{2}x^2\right)$
34. $f(x) = x^2 \ln(1 + x^3)$
35. $f(x) = \dfrac{x}{\sqrt{4 + x^2}}$
36. $f(x) = \dfrac{x^2}{\sqrt{2 + x}}$
37. $f(x) = \sin^2 x$ $\left[\textit{Hint:} \text{ Use } \sin^2 x = \frac{1}{2}(1 - \cos 2x).\right]$
38. $f(x) = \begin{cases} \dfrac{x - \sin x}{x^3} & \text{if } x \neq 0 \\ \frac{1}{6} & \text{if } x = 0 \end{cases}$

39–42 Find the Maclaurin series of f (by any method) and its radius of convergence. Graph f and its first few Taylor polynomials on the same screen. What do you notice about the relationship between these polynomials and f?

39. $f(x) = \cos(x^2)$
40. $f(x) = e^{-x^2} + \cos x$
41. $f(x) = xe^{-x}$
42. $f(x) = \tan^{-1}(x^3)$

43. Use the Maclaurin series for $\cos x$ to compute $\cos 5°$ correct to five decimal places.

44. Use the Maclaurin series for e^x to calculate $1/\sqrt[10]{e}$ correct to five decimal places.

45. (a) Use the binomial series to expand $1/\sqrt{1 - x^2}$.
(b) Use part (a) to find the Maclaurin series for $\sin^{-1}x$.