11.2 Exercises

1. (a) What is the difference between a sequence and a series?
   (b) What is a convergent series? What is a divergent series?

2. Explain what it means to say that $\sum_{n=1}^{\infty} a_n = 5$.

3–4 Calculate the sum of the series $\sum_{n=1}^{\infty} a_n$ whose partial sums are given.

3. $s_n = 2 - 3(0.8)^n$
4. $s_n = \dfrac{n^2 - 1}{4n^2 + 1}$

5–8 Calculate the first eight terms of the sequence of partial sums correct to four decimal places. Does it appear that the series is convergent or divergent?

5. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$
6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}$
7. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{1+\sqrt{n}}$
8. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!}$

9–14 Find at least 10 partial sums of the series. Graph both the sequence of terms and the sequence of partial sums on the same screen. Does it appear that the series is convergent or divergent? If it is convergent, find the sum. If it is divergent, explain why.

9. $\displaystyle\sum_{n=1}^{\infty} \frac{12}{(-5)^n}$
10. $\displaystyle\sum_{n=1}^{\infty} \cos n$
11. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+4}}$
12. $\displaystyle\sum_{n=1}^{\infty} \frac{7^{n+1}}{10^n}$
13. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right)$
14. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(n+2)}$

15. Let $a_n = \dfrac{2n}{3n+1}$.
    (a) Determine whether $\{a_n\}$ is convergent.
    (b) Determine whether $\sum_{n=1}^{\infty} a_n$ is convergent.

16. (a) Explain the difference between
$$\sum_{i=1}^{n} a_i \quad \text{and} \quad \sum_{j=1}^{n} a_j$$
    (b) Explain the difference between
$$\sum_{i=1}^{n} a_i \quad \text{and} \quad \sum_{i=1}^{n} a_j$$

17–26 Determine whether the geometric series is convergent or divergent. If it is convergent, find its sum.

17. $3 - 4 + \frac{16}{3} - \frac{64}{9} + \cdots$
18. $4 + 3 + \frac{9}{4} + \frac{27}{16} + \cdots$
19. $10 - 2 + 0.4 - 0.08 + \cdots$
20. $2 + 0.5 + 0.125 + 0.03125 + \cdots$
21. $\displaystyle\sum_{n=1}^{\infty} 6(0.9)^{n-1}$
22. $\displaystyle\sum_{n=1}^{\infty} \frac{10^n}{(-9)^{n-1}}$
23. $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n}$
24. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(\sqrt{2})^n}$
25. $\displaystyle\sum_{n=0}^{\infty} \frac{\pi^n}{3^{n+1}}$
26. $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{3^{n-1}}$

27–42 Determine whether the series is convergent or divergent. If it is convergent, find its sum.

27. $\frac{1}{3} + \frac{1}{6} + \frac{1}{9} + \frac{1}{12} + \frac{1}{15} + \cdots$
28. $\frac{1}{3} + \frac{2}{9} + \frac{1}{27} + \frac{2}{81} + \frac{1}{243} + \frac{2}{729} + \cdots$
29. $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{3n-1}$
30. $\displaystyle\sum_{k=1}^{\infty} \frac{k(k+2)}{(k+3)^2}$
31. $\displaystyle\sum_{n=1}^{\infty} \frac{1+2^n}{3^n}$
32. $\displaystyle\sum_{n=1}^{\infty} \frac{1+3^n}{2^n}$
33. $\displaystyle\sum_{n=1}^{\infty} \sqrt[n]{2}$
34. $\displaystyle\sum_{n=1}^{\infty} [(0.8)^{n-1} - (0.3)^n]$
35. $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n^2+1}{2n^2+1}\right)$
36. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+(\frac{2}{3})^n}$
37. $\displaystyle\sum_{k=0}^{\infty} \left(\frac{\pi}{3}\right)^k$
38. $\displaystyle\sum_{k=1}^{\infty} (\cos 1)^k$
39. $\displaystyle\sum_{n=1}^{\infty} \arctan n$
40. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{5^n} + \frac{2}{n}\right)$
41. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{e^n} + \frac{1}{n(n+1)}\right)$
42. $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n^2}$

43–48 Determine whether the series is convergent or divergent by expressing s_n as a telescoping sum (as in Example 7). If it is convergent, find its sum.

43. $\displaystyle\sum_{n=2}^{\infty} \frac{2}{n^2-1}$
44. $\displaystyle\sum_{n=1}^{\infty} \ln\frac{n}{n+1}$
45. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$
46. $\displaystyle\sum_{n=1}^{\infty} \left(\cos\frac{1}{n^2} - \cos\frac{1}{(n+1)^2}\right)$
47. $\displaystyle\sum_{n=1}^{\infty} (e^{1/n} - e^{1/(n+1)})$
48. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^3-n}$

49. Let $x = 0.99999\ldots$.
(a) Do you think that $x < 1$ or $x = 1$?
(b) Sum a geometric series to find the value of x.
(c) How many decimal representations does the number 1 have?
(d) Which numbers have more than one decimal representation?

50. A sequence of terms is defined by

$$a_1 = 1 \qquad a_n = (5 - n)a_{n-1}$$

Calculate $\sum_{n=1}^{\infty} a_n$.

51–56 Express the number as a ratio of integers.

51. $0.\overline{8} = 0.8888\ldots$
52. $0.\overline{46} = 0.46464646\ldots$
53. $2.\overline{516} = 2.516516516\ldots$
54. $10.1\overline{35} = 10.135353535\ldots$
55. $1.53\overline{42}$
56. $7.\overline{12345}$

57–63 Find the values of x for which the series converges. Find the sum of the series for those values of x.

57. $\sum_{n=1}^{\infty} (-5)^n x^n$
58. $\sum_{n=1}^{\infty} (x+2)^n$
59. $\sum_{n=0}^{\infty} \frac{(x-2)^n}{3^n}$
60. $\sum_{n=0}^{\infty} (-4)^n (x-5)^n$
61. $\sum_{n=0}^{\infty} \frac{2^n}{x^n}$
62. $\sum_{n=0}^{\infty} \frac{\sin^n x}{3^n}$
63. $\sum_{n=0}^{\infty} e^{nx}$

64. We have seen that the harmonic series is a divergent series whose terms approach 0. Show that

$$\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n}\right)$$

is another series with this property.

CAS **65–66** Use the partial fraction command on your CAS to find a convenient expression for the partial sum, and then use this expression to find the sum of the series. Check your answer by using the CAS to sum the series directly.

65. $\sum_{n=1}^{\infty} \frac{3n^2 + 3n + 1}{(n^2 + n)^3}$
66. $\sum_{n=3}^{\infty} \frac{1}{n^5 - 5n^3 + 4n}$

67. If the nth partial sum of a series $\sum_{n=1}^{\infty} a_n$ is

$$s_n = \frac{n-1}{n+1}$$

find a_n and $\sum_{n=1}^{\infty} a_n$.

68. If the nth partial sum of a series $\sum_{n=1}^{\infty} a_n$ is $s_n = 3 - n2^{-n}$, find a_n and $\sum_{n=1}^{\infty} a_n$.

69. A patient takes 150 mg of a drug at the same time every day. Just before each tablet is taken, 5% of the drug remains in the body.
(a) What quantity of the drug is in the body after the third tablet? After the nth tablet?
(b) What quantity of the drug remains in the body in the long run?

70. After injection of a dose D of insulin, the concentration of insulin in a patient's system decays exponentially and so it can be written as De^{-at}, where t represents time in hours and a is a positive constant.
(a) If a dose D is injected every T hours, write an expression for the sum of the residual concentrations just before the $(n + 1)$st injection.
(b) Determine the limiting pre-injection concentration.
(c) If the concentration of insulin must always remain at or above a critical value C, determine a minimal dosage D in terms of C, a, and T.

71. When money is spent on goods and services, those who receive the money also spend some of it. The people receiving some of the twice-spent money will spend some of that, and so on. Economists call this chain reaction the *multiplier effect*. In a hypothetical isolated community, the local government begins the process by spending D dollars. Suppose that each recipient of spent money spends $100c\%$ and saves $100s\%$ of the money that he or she receives. The values c and s are called the *marginal propensity to consume* and the *marginal propensity to save* and, of course, $c + s = 1$.
(a) Let S_n be the total spending that has been generated after n transactions. Find an equation for S_n.
(b) Show that $\lim_{n\to\infty} S_n = kD$, where $k = 1/s$. The number k is called the *multiplier*. What is the multiplier if the marginal propensity to consume is 80%?

Note: The federal government uses this principle to justify deficit spending. Banks use this principle to justify lending a large percentage of the money that they receive in deposits.

72. A certain ball has the property that each time it falls from a height h onto a hard, level surface, it rebounds to a height rh, where $0 < r < 1$. Suppose that the ball is dropped from an initial height of H meters.
(a) Assuming that the ball continues to bounce indefinitely, find the total distance that it travels.
(b) Calculate the total time that the ball travels. (Use the fact that the ball falls $\frac{1}{2}gt^2$ meters in t seconds.)
(c) Suppose that each time the ball strikes the surface with velocity v it rebounds with velocity $-kv$, where $0 < k < 1$. How long will it take for the ball to come to rest?

73. Find the value of c if

$$\sum_{n=2}^{\infty} (1 + c)^{-n} = 2$$

74. Find the value of c such that

$$\sum_{n=0}^{\infty} e^{nc} = 10$$

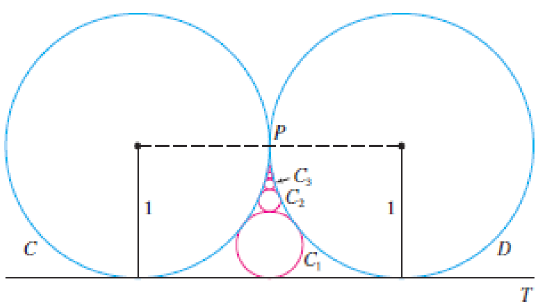
75. In Example 8 we showed that the harmonic series is divergent. Here we outline another method, making use of the fact that $e^x > 1 + x$ for any $x > 0$. (See Exercise 6.2.103.)

If s_n is the nth partial sum of the harmonic series, show that $e^{s_n} > n + 1$. Why does this imply that the harmonic series is divergent?

76. Graph the curves $y = x^n$, $0 \leqslant x \leqslant 1$, for $n = 0, 1, 2, 3, 4, \ldots$ on a common screen. By finding the areas between successive curves, give a geometric demonstration of the fact, shown in Example 7, that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

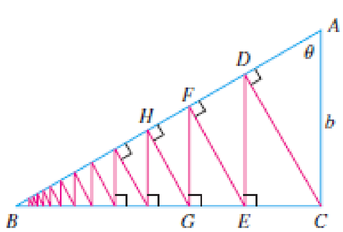
77. The figure shows two circles C and D of radius 1 that touch at P. T is a common tangent line; C_1 is the circle that touches C, D, and T; C_2 is the circle that touches C, D, and C_1; C_3 is the circle that touches C, D, and C_2. This procedure can be continued indefinitely and produces an infinite sequence of circles $\{C_n\}$. Find an expression for the diameter of C_n and thus provide another geometric demonstration of Example 7.



78. A right triangle ABC is given with $\angle A = \theta$ and $|AC| = b$. CD is drawn perpendicular to AB, DE is drawn perpendicular to BC, $EF \perp AB$, and this process is continued indefinitely, as shown in the figure. Find the total length of all the perpendiculars

$$|CD| + |DE| + |EF| + |FG| + \cdots$$

in terms of b and θ.



79. What is wrong with the following calculation?

$$\begin{aligned} 0 &= 0 + 0 + 0 + \cdots \\ &= (1 - 1) + (1 - 1) + (1 - 1) + \cdots \\ &= 1 - 1 + 1 - 1 + 1 - 1 + \cdots \\ &= 1 + (-1 + 1) + (-1 + 1) + (-1 + 1) + \cdots \\ &= 1 + 0 + 0 + 0 + \cdots = 1 \end{aligned}$$

(Guido Ubaldus thought that this proved the existence of God because "something has been created out of nothing.")

80. Suppose that $\sum_{n=1}^{\infty} a_n$ $(a_n \neq 0)$ is known to be a convergent series. Prove that $\sum_{n=1}^{\infty} 1/a_n$ is a divergent series.

81. Prove part (i) of Theorem 8.

82. If $\Sigma\, a_n$ is divergent and $c \neq 0$, show that $\Sigma\, ca_n$ is divergent.

83. If $\Sigma\, a_n$ is convergent and $\Sigma\, b_n$ is divergent, show that the series $\Sigma\, (a_n + b_n)$ is divergent. [*Hint:* Argue by contradiction.]

84. If $\Sigma\, a_n$ and $\Sigma\, b_n$ are both divergent, is $\Sigma\, (a_n + b_n)$ necessarily divergent?

85. Suppose that a series $\Sigma\, a_n$ has positive terms and its partial sums s_n satisfy the inequality $s_n \leqslant 1000$ for all n. Explain why $\Sigma\, a_n$ must be convergent.

86. The Fibonacci sequence was defined in Section 11.1 by the equations

$$f_1 = 1, \quad f_2 = 1, \quad f_n = f_{n-1} + f_{n-2} \qquad n \geqslant 3$$

Show that each of the following statements is true.

(a) $\dfrac{1}{f_{n-1} f_{n+1}} = \dfrac{1}{f_{n-1} f_n} - \dfrac{1}{f_n f_{n+1}}$

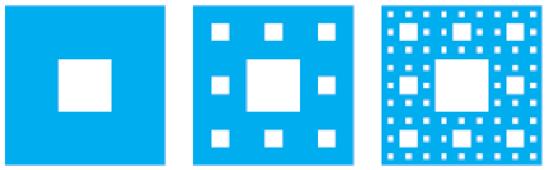
(b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{f_{n-1} f_{n+1}} = 1$

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{f_n}{f_{n-1} f_{n+1}} = 2$

87. The **Cantor set**, named after the German mathematician Georg Cantor (1845–1918), is constructed as follows. We start with the closed interval $[0, 1]$ and remove the open interval $(\frac{1}{3}, \frac{2}{3})$. That leaves the two intervals $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$ and we remove the open middle third of each. Four intervals remain and again we remove the open middle third of each of them. We continue this procedure indefinitely, at each step removing the open middle third of every interval that remains from the preceding step. The Cantor set consists of the numbers that remain in $[0, 1]$ after all those intervals have been removed.

(a) Show that the total length of all the intervals that are removed is 1. Despite that, the Cantor set contains infinitely many numbers. Give examples of some numbers in the Cantor set.

(b) The **Sierpinski carpet** is a two-dimensional counterpart of the Cantor set. It is constructed by removing the center one-ninth of a square of side 1, then removing the centers of the eight smaller remaining squares, and so on. (The figure shows the first three steps of the construction.) Show that the sum of the areas of the removed squares is 1. This implies that the Sierpinski carpet has area 0.



88. (a) A sequence $\{a_n\}$ is defined recursively by the equation $a_n = \frac{1}{2}(a_{n-1} + a_{n-2})$ for $n \geqslant 3$, where a_1 and a_2 can be any real numbers. Experiment with various values of a_1 and a_2 and use your calculator to guess the limit of the sequence.
(b) Find $\lim_{n\to\infty} a_n$ in terms of a_1 and a_2 by expressing $a_{n+1} - a_n$ in terms of $a_2 - a_1$ and summing a series.

89. Consider the series $\sum_{n=1}^{\infty} n/(n+1)!$.
(a) Find the partial sums s_1, s_2, s_3, and s_4. Do you recognize the denominators? Use the pattern to guess a formula for s_n.
(b) Use mathematical induction to prove your guess.
(c) Show that the given infinite series is convergent, and find its sum.

90. In the figure there are infinitely many circles approaching the vertices of an equilateral triangle, each circle touching other circles and sides of the triangle. If the triangle has sides of length 1, find the total area occupied by the circles.